6.041 Spring 2011 Quiz 1  
Monday, March 7, 12:05 - 12:55 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: __________________________
Recitation Instructor: ________________________
TA: __________________________

<table>
<thead>
<tr>
<th>Question</th>
<th>Score</th>
<th>Out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (a)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>1 (b)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>1 (c)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>2 (a)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>2 (b)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>2 (c)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>3 (a)</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>3 (b)</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>3 (c)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>Your Grade</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

- This quiz has 3 problems, worth a total of 100 points.
- You may tear apart pages 3 and 4, as per your convenience, **but you must turn them in together with the rest of the booklet**.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can’t read it.
- You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- **You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator.** Expressions like \( \binom{8}{3} \) or \( \sum_{k=0}^{5}(1/2)^k \) are also fine.
- You have 50 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Tuesday 3/8.
Problem 0: (0 points) Write your name, your assigned recitation instructor’s name, and assigned TA’s name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

<table>
<thead>
<tr>
<th>Recitation Instructor</th>
<th>TA</th>
<th>Recitation Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mardavij Roozbehani</td>
<td>Wendy Gu</td>
<td>10 &amp; 11 AM</td>
</tr>
<tr>
<td>John Fisher</td>
<td>Andrew Mastin</td>
<td>1 &amp; 2 PM</td>
</tr>
<tr>
<td>Shan-Yuan Ho (6.431)</td>
<td>Weifei Zeng</td>
<td>12 noon</td>
</tr>
</tbody>
</table>
Problem 1. (36 points) The probability that a person selected at random has a certain disease is \( \frac{1}{N} \). Two partners in a medical practice use different tests for diagnosis of this disease. Dr. Jones, uses test \( T_A \) which yields a positive result with probability \( \frac{m-1}{m} \) if the patient actually has the disease. Dr. Smith uses two tests, \( T_B \) and \( T_C \), which each yield a positive result with probability \( \sqrt{\frac{m-1}{m}} \) if the patient has the disease. The results of tests \( T_B \) and \( T_C \) are independent if the patient has the disease. They are also independent if the patient does not have the disease. The false positive rate for test \( T_B \) is \( b \) while the false positive rate for test \( T_C \) is \( 4b \).

Assume that \( N, m > 1, 0 < b < \frac{1}{4} \) and that \( N \) is much larger than \( m \). Let

- \( A \) be the event that \( T_A \) gives a positive result.
- \( B \) be the event that \( T_B \) gives a positive result.
- \( C \) be the event that \( T_C \) gives a positive result.
- \( D \) be the event that a person has the disease.

(a) (12 points) Dr. Jones states that if test \( T_A \) has a positive test result, the probability that the patient actually has the disease equals \( \frac{m-1}{m} \). Find an expression for the false positive rate for test \( T_A \) in terms of \( N \) and \( m \) such that this is correct.

(b) (12 points) Dr. Smith uses both tests \( T_B \) and \( T_C \), and gives a positive diagnosis only if both results are positive. Find an expression for \( b \) in terms of \( N \) and \( m \) such that the probability that the patient has the disease conditioned on positive results for both \( T_B \) and \( T_C \) is also \( \frac{m-1}{m} \).

(c) (12 points) If \( T_B \) is positive and \( T_C \) is negative find an expression for the probability that the patient does not have the disease using only the terms \( P(D) \), \( P(\bar{D}) \), \( P(B|D) \), \( P(B|\bar{D}) \), \( P(C|D) \), and \( P(C|\bar{D}) \) (i.e. not in terms of \( N \), \( m \), or \( b \)).

Problem 2. (36 points) Please refer Figure 1 on page 4. A manufacturing company has \( m \) facilities. Each facility \( w_i \), \( i = 1, ..., m \) has a fast production machine of type-B, and a slow production machine of type-C, and can continue to produce (stay online) as long as either the type-B or the type-C machine is working. Otherwise, it will go offline. Let \( B_i \) and \( C_i \) be independent random variables denoting, respectively, the lifetime (in days) of type-B and type-C machines in \( w_i \). We have for all \( i = 1, ..., m \):

\[
p_{B_i}(b) = \begin{cases} (2^{0.2} - 1)2^{-0.1b} & , \quad b = 1, 2, ... \\ 0 & , \quad otherwise \end{cases} \quad p_{C_i}(c) = \begin{cases} (2^{0.01} - 1)2^{-0.01c} & , \quad c = 1, 2, ... \\ 0 & , \quad otherwise \end{cases}
\]

(a) (12 points) Given \( t \in \{1, 2, ...\} \), what is the probability that the life of machine of type-B in \( w_1 \) exceeds \( t \) days, i.e., \( B_1 > t \)?

(b) (12 points) Given that machine of type-B in \( w_3 \) has been working for \( t \) days now, what is its expected remaining lifetime? Show your work clearly.

(c) (12 points) At some point in the future, we find out that a total of \( a \) machines are broken. Suppose that each broken machine is equally likely to be a type-B or type-C. What is the
expected number of facilities that are online? Show your work clearly and justify your steps based on assumptions.

Figure 1: Manufacturing company in Problem 2

Problem 3. (28 points) Ann and her friend Gary are trading baseball cards. To get a complete collection of cards, a person needs to have 10 different kinds of cards. To start out with, each person has 10 cards. Each of those cards has probability \( \frac{1}{10} \) of being of any particular kind and each card of any person is of a kind independently from any other card of any person.

(a) (8 points) What is the probability that Ann has a complete collection to start out with?

(b) (8 points) What is the probability that both Ann and Gary have a complete collection to start out with?

(c) (12 points) Suppose Ann does not start out with a complete set. Gary will agree to trade one card with Ann, only if Ann gives him in return a card he did not already have. What is the probability that Ann ends up with a complete set, conditioned on the event that Ann does not start out with a complete set?

Each question is repeated in the following pages. Please write your answer on the appropriate page.
Problem 1. (36 points) The probability that a person selected at random has a certain disease is \( \frac{1}{N} \). Two partners in a medical practice use different tests for diagnosis of this disease. Dr. Jones uses test \( T_A \) which yields a positive result with probability \( \frac{m-1}{m} \) if the patient actually has the disease. Dr. Smith uses two tests, \( T_B \) and \( T_C \), which each yield a positive result with probability \( \sqrt{\frac{m-1}{m}} \) if the patient has the disease. The results of tests \( T_B \) and \( T_C \) are independent if the patient has the disease. They are also independent if the patient does not have the disease. The false positive rate for test \( T_B \) is \( b \) while the false positive rate for test \( T_C \) is \( 4b \).

Assume that \( N, m > 1, 0 < b < \frac{1}{4} \) and that \( N \) is much larger than \( m \).

Let
- \( A \) be the event that \( T_A \) gives a positive result.
- \( B \) be the event that \( T_B \) gives a positive result.
- \( C \) be the event that \( T_C \) gives a positive result.
- \( D \) be the event that a person has the disease.

(a) (12 points) Dr. Jones states that if test \( T_A \) has a positive test result, the probability that the patient actually has the disease equal to \( \frac{m-1}{m} \). Find an expression for the false positive rate for test \( T_A \) in terms of \( N \) and \( m \) such that this is correct.
(b) **(12 points)** Dr. Smith uses both tests $T_B$ and $T_C$, and gives a positive diagnosis only if both results are positive. Find an expression for $b$ in terms of $N$ and $m$ such that the probability that the patient has the disease conditioned on positive results for both $T_B$ and $T_C$ is also $\frac{m-1}{m}$. 
(c) **(12 points)** If $T_B$ is positive and $T_C$ is negative find an expression for the probability that the patient does not have the disease using only the terms $P(D)$, $P(\bar{D})$, $P(B|D)$, $P(B|\bar{D})$, $P(C|D)$, and $P(C|\bar{D})$ (i.e. **not** in terms of $N$, $m$, or $b$).
Problem 2. (36 points) Please refer Figure 1 on page 4. A manufacturing company has $m$ facilities. Each facility $w_i$, $i = 1, ..., m$ has a fast production machine of type-B, and a slow production machine of type-C, and can continue to produce (stay online) as long as either the type-B or the type-C machine is working. Otherwise, it will go offline. Let $B_i$ and $C_i$ be independent random variables denoting, respectively, the lifetime (in days) of type-B and type-C machines in $w_i$. We have for all $i = 1, ..., m$:

$$
p_{B_i}(b) = \begin{cases} (2^{0.1} - 1)2^{-0.1b}, & b = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}, \quad p_{C_i}(c) = \begin{cases} (2^{0.01} - 1)2^{-0.01c}, & c = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}
$$

(a) (12 points) Given $t \in \{1, 2, \ldots\}$, what is the probability that the life of machine of type-B in $w_1$ exceeds $t$ days, i.e., $B_1 > t$?
(b) \textbf{(12 points)} Given that machine of type-B in \( w_3 \) has been working for \( t \) days now, what is its expected remaining lifetime? Show your work clearly.
(c) (12 points) At some point in the future, we find out that a total of $a$ machines are broken. Suppose that each broken machine is equally likely to be a type-B or type-C. What is the expected number of facilities that are online? Show your work clearly and justify your steps based on assumptions.
Problem 3. (28 points) Ann and her friend Gary are trading baseball cards. To get a complete collection of cards, a person needs to have 10 different kinds of cards.

To start out with, each person has 10 cards. Each of those cards has probability \( \frac{1}{10} \) of being of any particular kind and each card of any person is of a kind independently from any other card of any person.

(a) (8 points) What is the probability that Ann has a complete collection to start out with?

(b) (8 points) What is the probability that both Ann and Gary have a complete collection to start out with?
(c) **(12 points)** Suppose Ann does not start out with a complete set. Gary will agree to trade one card with Ann, only if Ann gives him in return a card he did not already have. What is the probability that Ann ends up with a complete set, conditioned on the event that Ann does not start out with a complete set?