6.431 Spring 2011 Quiz 1
Monday, March 7, 12:05 - 12:55 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO
MAKE SURE YOU GET THE RIGHT PAPER. THIS IS FOR 6.431

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<tr>
<th>Question</th>
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<td>3 (f)</td>
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<td>Your Grade</td>
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Name: ____________________________
Recitation Instructor: ____________________________
TA: ____________________________

- This quiz has 3 problems, worth a total of 100 points.
- You may tear apart pages 3 and 4, as per your convenience, but you must turn them in together with the rest of the booklet.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can’t read it.
- You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like \( \binom{5}{3} \) or \( \sum_{k=0}^{5}(1/2)^k \) are also fine.
- You have 50 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Tuesday 3/8.
Problem 0: (0 points) Write your name, your assigned recitation instructor’s name, and assigned TA’s name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

<table>
<thead>
<tr>
<th>Recitation Instructor</th>
<th>TA</th>
<th>Recitation Time</th>
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<tbody>
<tr>
<td>Mardavij Roozbehani</td>
<td>Wendy Gu</td>
<td>10 &amp; 11 AM</td>
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<td>John Fisher</td>
<td>Andrew Mastin</td>
<td>1 &amp; 2 PM</td>
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<tr>
<td>Shan-Yuan Ho (6.431)</td>
<td>Weifei Zeng</td>
<td>12 noon</td>
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</tbody>
</table>
Problem 1. (10 points) Urn 1 contains 3 white balls and 5 black balls. Urn 2 contains 4 white balls and 2 black balls. Urn 3 contains 2 white balls and no black balls. An urn is selected at random and 2 balls are randomly chosen from the selected urn. What is the probability that the balls came from Urn 1 given that there was one black and one white ball selected?

Problem 2. (36 points) There are \( m \) facilities in a manufacturing plant. Each facility \( w_i, i = 1, \ldots, m \), has a fast production machine of type-B and a slow production machine of type-C. A facility will stay online if either machine (type-B or type-C) is currently working. Otherwise, it will go offline. Each machine is independent of other machines and each facility is independent of other facilities. For each \( w_i \), the lifetime (in days) of the type-B and type-C machines are random variables \( B_i \) and \( C_i \), respectively, described by the following PMFs.

\[
p_{B_i}(b) = \begin{cases} 
(2^{0.1} - 1)2^{-0.1b}, & b = 1, 2, \ldots \\
0, & \text{otherwise}
\end{cases}, \quad p_{C_i}(c) = \begin{cases} 
(2^{0.01} - 1)2^{-0.01c}, & c = 1, 2, \ldots \\
0, & \text{otherwise}
\end{cases}
\]

(a) (12 points) What is the probability that the life of machine type-B in \( w_1 \) exceeds \( t \) days?

(b) (12 points) Given that the machine of type-B on \( w_3 \) has been working for \( t \) days, what is its expected remaining lifetime? Show your work clearly.

(c) (12 points) At some point in the future, we find out that a total of \( a \) machines are broken. Suppose that each broken machine is equally likely to be a type-B or type-C. What is the expected number of facilities that are online? Show your work clearly and justify your steps based on assumptions.

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Figure 1: Manufacturing company in Problem 2
Problem 3. (54 points) Ann and her friend Gary are trading baseball cards. To get a complete collection of cards, a person needs to have 10 different kinds of cards.
To start out with, each person has 10 cards. Each of those cards has probability $\frac{1}{10}$ of being of any particular kind and each card of any person is of a kind independently from any other card of any person.

(a) (8 points) What is the probability that Ann has a complete collection to start out with?

(b) (4 points) What is the probability that both Ann and Gary have a complete collection to start out with?

(c) (16 points) Suppose Ann does not start out with a complete set. Gary will agree to trade one card with Ann, only if Ann gives him in return a card he did not already have. What is the probability that Ann ends up with a complete set, conditioned on the event that Ann does not start out with a complete set?

(d) (8 points) Assume that Ann starts out with zero cards and obtains one card at a time. Every card she gets is equally likely to be any of the 10 kinds of cards. What is the expected number of cards she needs to get in order to obtain a complete collection of cards?

(e) (6 points) Suppose Ann has 10 cards of 2 types, 5 of the same card and 5 of another card. Gary will randomly choose 2 cards from Ann’s set. Gary gets $2.00 if the cards are different and -$1.00 (lose $1.00) if the cards are the same. Find the expected value and variance of Gary’s winning’s.

(f) (12 points) Suppose that Ann has only 2 types of cards, $n$ cards of Type 1 and $m$ cards of type 2, in her hand of 10 cards. The cards are randomly removed one at a time until only those of one kind are left. Find the probability that only cards of Type 1 are left.

Each question is repeated in the following pages. Please write your answer on the appropriate page.
Problem 1. (10 points) Urn 1 contains 3 white balls and 5 black balls. Urn 2 contains 4 white balls and 2 black balls. Urn 3 contains 2 white balls and no black balls. An urn is selected at random and 2 balls are randomly chosen from the selected urn. What is the probability that the balls came from Urn 1 given that there was one black and one white ball selected?
(36 points) There are $m$ facilities in a manufacturing plant. Each facility $w_i$, $i = 1, ..., m$, has a fast production machine of type-B and a slow production machine of type-C. A facility will stay online if either machine (type-B or type-C) is currently working. Otherwise, it will go offline. Each machine is independent of other machines and each facility is independent of other facilities. For each $w_i$, the lifetime (in days) of the type-B and type-C machines are random variables $B_i$ and $C_i$, respectively, described by the following PMFs.

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0, & \text{otherwise}
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0, & \text{otherwise}
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(a) (12 points) What is the probability that the life of machine type-B in $w_1$ exceeds $t$ days?
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(c) (12 points) At some point in the future, we find out that a total of $a$ machines are broken. Suppose that each broken machine is equally likely to be a type-B or type-C. What is the expected number of facilities that are online? Show your work clearly and justify your steps based on assumptions.
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