Problem Set 7 Solutions

**Exercise 7-1.** Do Exercise 29.2–5 on p. 864.

**Exercise 7-2.** Do Exercise 29.2–6 on p. 864.

**Exercise 7-3.** Do Exercise 29.2–7 on p. 864.

**Exercise 7-4.** Do Exercise 29.4–3 on p. 885.

**Exercise 7-5.** Do Exercise 29.4–5 on p. 885.
Problem 7-1. Solving linear programs.

Consider the following linear program:

\[
\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad 3x_1 + x_2 \leq 3 \\
& \quad x_1 + 3x_2 \leq 3 \\
& \quad x_1 + x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(a) Find the polygon that corresponds to the feasible region defined by the constraints and solve the linear program.

**Solution:** The polygon that corresponds to the feasible region is the polygon with vertices \((x_1, x_2) = (0, 0), (1, 0), (0, 1), (3/4, 3/4)\). The optimal solution lies on one of the vertices. We simply test all vertices and find that \((3/4, 3/4)\) is the optimal solution with objective value \(3/2\).

(b) Run the Simplex algorithm on the linear program.

**Solution:** We convert the linear program into the following equivalent form in order to use the Simplex algorithm presented in lecture and in the handout

\[
\begin{align*}
\text{minimize} & \quad -x_1 - x_2 \\
\text{subject to} & \quad -3x_1 - x_2 \geq -3 \\
& \quad -x_1 - 3x_2 \geq -3 \\
& \quad -x_1 - x_2 \geq -2 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

The maximization problem was changed to a minimization one by taking the negative of the objective value, and the constraints were all written with \(\geq\). We now introduce the basic variables \(y_1, y_2, y_3 \geq 0\).
Iteration 0

\[
\begin{align*}
\text{minimize} & \quad -x_1 - x_2 \\
\text{subject to} & \quad y_1 = 3 - 3x_1 - x_2 \\
& \quad y_2 = 3 - x_1 - 3x_2 \\
& \quad y_3 = 2 - x_1 - x_2 \\
& \quad x_1, x_2, y_1, y_2, y_3 \geq 0.
\end{align*}
\]

Pick one of variables with a negative coefficient in the objective function as the next basic variable. We’ll choose \( x_1 \). There are three ways to make \( x_1 \) a basic variable.

1. Substitute \( x_1 = 1 - y_1/3 - x_2/3 \)
2. Substitute \( x_1 = 3 - y_2 - 3x_2 \)
3. Substitute \( x_1 = 2 - y_3 - x_2 \)

We pick the first equation because it has the smallest non-negative free coefficient of 1.

Iteration 1

\[
\begin{align*}
\text{minimize} & \quad -1 + y_1/3 - 2x_2/3 \\
\text{subject to} & \quad x_1 = 1 - y_1/3 - x_2/3 \\
& \quad y_2 = 2 + y_1/3 - 8x_2/3 \\
& \quad y_3 = 1 + y_1/3 - 2x_2/3 \\
& \quad x_1, x_2, y_1, y_2, y_3 \geq 0.
\end{align*}
\]

We pick \( x_2 \) as the next basic variable. There are three ways to make \( x_2 \) a basic variable.

1. Substitute \( x_2 = 3 - y_1 - 3x_2 \).
2. Substitute \( x_2 = 3/4 + y_1/8 - 3y_2/8 \).
3. Substitute \( x_2 = 3/2 + y_1/2 - 3y_3/2 \).

We pick the second equation because it has the smallest non-negative free coefficient of 3/4.

Iteration 2

\[
\begin{align*}
\text{minimize} & \quad -3/2 + y_1/4 + y_2/4 \\
\text{subject to} & \quad x_1 = 3/4 - 9y_1/24 + y_2/8 \\
& \quad x_2 = 3/4 + y_1/8 - 3y_2/8 \\
& \quad y_3 = 1/2 + y_1/4 + y_2/4 \\
& \quad x_1, x_2, y_1, y_2, y_3 \geq 0.
\end{align*}
\]
There are no more variables in the objective function with negative coefficients, so we’re done, and the current basic feasible solution of \( x_1 = 3/4, x_2 = 3/4, y_3 = 1/2, y_1 = 0, y_2 = 0 \) is optimal. Note that we get the same solution \((x_1, x_2) = (3/4, 3/4)\) as in part (a).

**Problem 7-2. Reducing linear programming to the feasibility problem.**

Suppose we have an algorithm \( A \) that solves the feasibility problem, i.e., given constraints of the form

\[
\begin{align*}
A_1 x & \leq b_1 \\
x & \geq 0
\end{align*}
\]

where \( A_1 \) is an \( m_1 \times n_1 \) matrix, \( x \) is an \( n_1 \)-vector, and \( b_1 \) is an \( m_1 \)-vector, \( A \) finds an \( x \) that satisfies the constraints, or declares that such does not exist.

Show how one can use \( A \) to solve a maximization problem, i.e., solve a linear program of the form

maximize

\[ c_2^T y \]

subject to

\[
\begin{align*}
A_2 y & \leq b_2 \\
y & \geq 0
\end{align*}
\]

where \( A_2 \) is an \( m_2 \times n_2 \) matrix, \( y \) is an \( n_2 \)-vector, \( b_2 \) is an \( m_2 \) vector, and \( c_2 \) is an \( n_2 \) vector. (*Hint:* Use duality.)

**Solution:** The dual of the linear program defined by (3)-(5) is

minimize

\[ b_2^T x \]

subject to

\[
\begin{align*}
A_2^T x & \geq c_2 \\
x & \geq 0
\end{align*}
\]

where \( x \) is a \( m_2 \)-vector. Consider a new linear program, denoted as \( LP \), that combines the constraints of both the primal and dual and includes the extra constraint that the objective values of the primal and dual are equal,
maximize

0

subject to

\[ A_2 y \leq b_2 \] (10)
\[ y \geq 0 \] (11)
\[ A_2^T x \geq c_2 \] (12)
\[ x \geq 0 \] (13)
\[ c_2^T y = b_2^T x \] (14)

The feasible solutions to \( LP \) can be seen as pairs \((\bar{x}, \bar{y})\). By construction, if \((\bar{x}, \bar{y})\) is a feasible solution to \( LP \), then \( \bar{x} \) is a feasible solution to the primal and \( \bar{y} \) to the dual. Moreover, the objective values of the primal and dual at \( \bar{x} \) and \( \bar{y} \), respectively, are equal to each other because of constraint (14). By the duality theorem, this equality only happens when \( \bar{x} \) and \( \bar{y} \) are the optimal solutions to their respective programs. Note that we’re only concerned about the constraints of \( LP \), so its objective value is irrelevant.

Therefore, if the primal has an optimal solution, we can find it from a feasible solution returned by running algorithm \( A \) on \( LP \). If the primal doesn’t have an optimal solution, then at least the primal or dual is infeasible, so \( A \) responds that no feasible solution to \( LP \) exists.