Quiz 1

- Do not open this quiz booklet until you are directed to do so. Read all the instructions first.
- The quiz contains 7 problems, several with multiple parts. You have 80 minutes to earn 80 points.
- This quiz booklet contains 9 pages, including this one, and a sheet of scratch paper which can be detached.
- This quiz is closed book. You may use one double sided Letter ($8\frac{1}{2}'' \times 11''$) or A4 crib sheet. No calculators or programmable devices are permitted. Cell phones must be put away.
- On page 2 there are several useful inequalities. Please review them before you start working on the quiz.
- Write your solutions in the space provided. If you run out of space, continue your answer on the back of the same sheet and make a notation on the front of the sheet.
- Do not waste time deriving facts that we have studied. It is sufficient to cite known results.
- Do not spend too much time on any one problem. Generally, a problem’s point value is an indication of how many minutes to spend on it.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Please be neat.
- Good luck!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Title</th>
<th>Points</th>
<th>Parts</th>
<th>Grade</th>
<th>Initials</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Name</td>
<td>1</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>True or False</td>
<td>21</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>S3L3CT</td>
<td>12</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>SWAT Team</td>
<td>12</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>FIFO = 2 × LIFO</td>
<td>12</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Big Edges</td>
<td>10</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Minimum Madness</td>
<td>12</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>80</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Name: __________________________
Problem 0. Name. [1 point] Write your name on every page of this exam booklet! Don’t forget the cover.

Possibly useful facts for elsewhere in the quiz

1. **Markov Inequality**: For any nonnegative random variable $X$, we have

   \[ \Pr \{ X \geq \lambda \} \leq \frac{\mathbb{E}[X]}{\lambda}. \]

2. **Chernoff Bounds**: Let $X_1, \ldots, X_n$ be $n$ independent Boolean random variables. Suppose that for $i = 1, 2, \ldots, n$, we have $\Pr \{ X_i = 1 \} = \delta_i$ for $0 \leq \delta_i \leq 1$. Let $X = \sum_{i=1}^{n} X_i$ and $\delta = (1/n) \cdot \sum_{i=1}^{n} \delta_i$. Then, for any $\delta \leq \gamma \leq 1$,

   \[ \Pr \{ X \geq \gamma n \} \leq e^{-2(\gamma - \delta)^2 n}. \]

   Note that this is a generalization of the Chernoff bound we saw in class to the case of not necessarily identically distributed random variables.

3. **Harmonic series**: \[
\sum_{i=1}^{n} \frac{1}{i} = \ln n + O(1).
\]
Problem 1. True or False. [21 points] (7 parts)
Circle T or F for each of the following statements to indicate whether the statement is true or false, respectively. You need not justify your answers, but since wrong answers will be penalized, do not guess unless you are reasonably sure.

(a) T F The recurrence $T(n) = 2T(\sqrt{n}) + \log n$ has solution $T(n) = \Theta(\log^2 n)$.

(b) T F The numbers 1, 2, . . . , 10 can be placed into a tree data structure such that the tree satisfies both the min-heap property and the binary-search-tree property at the same time.

(c) T F The following collection $\mathcal{H} = \{h_1, h_2, h_3\}$ of hash functions is universal, where each hash function maps the universe $U = \{A, B, C, D\}$ of keys into the range \{0, 1, 2\} according to the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>$h_1(x)$</th>
<th>$h_2(x)$</th>
<th>$h_3(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

(d) T F Consider a sequence of $n$ INSERT operations, $2n$ DECREASE-KEY operations, and $\sqrt{n}$ EXTRACT-MIN operations on an initially empty Fibonacci heap. The total running time of all these operations is $\Theta(n \log n)$ in the worst case.

(e) T F In the analysis of the disjoint-set data structure presented during lecture, once a node other than the root or a child of the root is block-charged, it will never again be path-charged.

(f) T F A van-Emde-Boas data structure can support FIND-MIN, FIND-MAX, SUCCESSOR, and PREDECESSOR operations over the set $\{1, 2, \ldots, 2\sqrt{n}\}$ in $O(\log n)$ time.

(g) T F Let $G = (V, E)$ be a connected undirected graph with edge-weight function $w : E \to \{1, 2, \ldots, 10 |E|\}$. Then a minimum spanning tree of $G$ can be constructed in $O(E \log \log V)$ time.
Problem 2. S3L3CT. [12 points]
Professor John Von Meanmann is implementing an algorithm to find the $i$th smallest of a set $S$ of $n$ distinct elements. The professor improvises on the traditional worst-case linear-time algorithm and organizes elements into groups of 3 instead of groups of 5, resulting in the following algorithm, called S3L3CT:

1. If $n = 1$, return the only element in the array.
2. Divide the $n$ elements of the input array into $\left\lfloor \frac{n}{3} \right\rfloor$ groups of 3 elements each, with 0 to 2 elements left over.
3. Find the median of each of the $\left\lceil \frac{n}{3} \right\rceil$ groups by rote.
4. Use S3L3CT recursively to find the median $x$ of the $\left\lceil \frac{n}{3} \right\rceil$ medians found in Step 3.
5. Partition the input array around $x$. Let $k = \text{RANK}(x)$.
6. If $i = k$, then return $x$. Otherwise, use S3L3CT recursively to find the $i$th smallest element on the low side if $i < k$, or the $(i - k)$th smallest element on the high side if $i > k$.

State the recurrence for the running time $T(n)$ of S3L3CT running on an input of $n$ elements, and provide a tight asymptotic upper bound on its solution in terms of $n$. In order to simplify the math, assume that any set on which S3L3CT operates contains a multiple of 3 elements.
**Problem 3. SWAT Team.** [12 points]

Two swatsmen with fly swatters are located at arbitrary positions along a long corridor with many leaky windows. One at a time, houseflies appear at various locations along the corridor, and a swatsman goes to the location of the fly and swats it dead. The cost of a given strategy is the total distance traveled by the swatsmen. Argue that the greedy strategy of the closest swatsman going to the location of the fly is not $\alpha$-competitive for any finite $\alpha$.

*(Hint: Consider the case that flies only appear at three locations $A$, $B$, and $C$, where $B$ falls between $A$ and $C$ and the distance from $A$ to $B$ is much smaller than the distance from $B$ to $C$, as shown below:)*

![Diagram](https://via.placeholder.com/150)

Consider the sequence $(C, A, B, A, B, A, B, A, B, A, \ldots)$ of fly arrivals.)
Problem 4. FIFO = 2 × LIFO. [12 points]
A FIFO queue $Q$ supporting the operations ENQUEUE and DEQUEUE can be implemented using two stacks $S_1$ and $S_2$, each of which supports the operations PUSH, POP, and a test whether the stack is empty.

ENQUEUE($Q, x$):
1. PUSH($S_1, x$)

DEQUEUE($Q$):
1. if $S_1 = \emptyset$ and $S_2 = \emptyset$
   2. error “queue underflow”
2. if $S_2 = \emptyset$
3. while $S_1 \neq \emptyset$
4. \[ x = \text{POP}(S_1) \]
5. \[ \text{PUSH}(S_2, x) \]
6. \[ \text{return POP}(S_2) \]

Define a potential function $\Phi(Q) = c |S_1|$ for an appropriate constant $c > 0$, where $|S_1|$ is the number of items in $S_1$. Argue using a potential-function argument that each ENQUEUE and DEQUEUE operation takes $O(1)$ amortized time.
Problem 5. Big Edges. [10 points]

Let $G = (V, E)$ be a connected undirected graph with distinct edge weights $w : E \to \mathbb{R}$, and let $c$ be a cycle in $G$. Consider the edge $e$ on $c$ with the largest weight, that is, $w(e) \geq w(e')$ for all $e' \in c$. Prove that $e$ does not belong to the minimum spanning tree of $G$. 

Problem 6. Minimum Madness. [12 points] (4 parts)
Consider the following program to find the minimum value in an array $A$ of $n$ distinct elements.

\begin{verbatim}
MINIMUM(A, n):
1 \hspace{1em} \text{\texttt{min}} = \infty \quad // \text{Set}\ \text{\texttt{min}}\ \text{to be a large value.}
2 \hspace{1em} \text{\texttt{for}}\ \text{\texttt{i}} = 1 \text{ to } n
3 \hspace{1em} \text{\texttt{if}}\ \text{\texttt{min}} > A[\text{\texttt{i}}]
4 \hspace{1em} \text{\texttt{min}} = A[\text{\texttt{i}}]
\end{verbatim}

Assume that $A$’s elements are randomly permuted before invoking $\text{MINIMUM}(A, n)$ and that all permutations are equally likely. Let $X_i$ be the indicator random variable associated with the event that the variable $\text{\texttt{min}}$ is changed in line 4 during the $i$th iteration of the \texttt{for} loop, and let $Y = \sum_{i=1}^{n} X_i$ be the random variable denoting the total number of times $\text{\texttt{min}}$ is so updated.

(a) Argue that the probability that $A[i]$ is smaller than all the elements in $A[1..i-1]$ is $1/i$.

(b) Show that $E[X_i] = 1/i$. 

(c) Prove that $E[Y] = \Theta(\lg n)$.

(d) Prove that for sufficiently large $n$, it holds that $\Pr \{Y \geq 5E[Y]\} \leq 1/n^4$. 