6.342 Lecture 7 — February 23, 2011

Today:
• Sampling and interpolation

Next time:
• More on sampling, including quantization effects

Schedule:
• PS2 posted today (almost surely) with a delayed due date
• Term project details posted tomorrow (almost surely)

Readings:
• Release α2.0 of textbook posted on fourierandwavelets.org
  – Since last Stellar posting: improvements mostly in Sections 1.5.5 and 1.6
  – Chapter numbering now matches course syllabus
• Review Chapter 3 material on your own
• Chapter 4 covered today and next time
• Supplements:

Sampling 101: Lowpass signals

\[ X(f) = \int x(t)e^{-j2\pi ft} dt \]

Whittaker–Kotel’nikov–Shannon sampling:
- **Nyquist rate** defined as \( f_{\text{Nyq}} = 2f_0 \)
- Uniform sampling at or above Nyquist rate **sufficient** for exact reconstruction.
- Reconstruct simple linear function of samples
Why does sampling work?

• In Signals & Systems:

No overlap between shifted spectra, ...

Why does sampling work?

• Lots of signals have the same samples

• Must restrict the set of signals such that the samples completely disambiguate any signal
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Figure 4.8: Sampling and interpolation for discrete-time sequences. (a) Sampling is filtering followed by downsampling, \( y_n = (\hat{h}_n * x_n)_{nN} \). (b) Interpolation is upsampling followed by filtering, \( \hat{x}_n = \sum_k y_k h_{n-kN} \).

**Definition 4.1 (Shift-invariant subspace of \( \ell^2(\mathbb{Z}) \))** A subspace \( W \subset \ell^2(\mathbb{Z}) \) is a shift-invariant subspace (SISS) with respect to shift \( L \in \mathbb{Z}^+ \) when \( x_n \in W \) implies \( x_{n-kL} \in W \) for every integer \( k \). In addition, \( w \in \ell^2(\mathbb{Z}) \) is called a generator of \( W \) when \( W = \text{span}(\{w_{n-kL}\}_{k \in \mathbb{Z}}) \).

Figure 4.10: Sampling and interpolation where the sampling prefilter and interpolation postfilter are related by time reversal.

**Proposition 4.2 (Sampling for SISS of \( \ell^2(\mathbb{Z}) \))** Consider the system shown in Figure 4.10 in which the sampling prefilter is the time-reversed version of the interpolation postfilter. Let \( S \) denote the SISS with respect to shift \( N \) generated by \( h \). If the filter satisfies (4.16), then:

1. Any input \( x \in S \) is reproduced perfectly, that is, \( \hat{x} = x \).
2. Any input \( x \not\in S \) is approximated optimally in the least-squares sense, that is, \( \|\hat{x} - x\|_2 \) is minimized among \( \hat{x} \in S \).

\[
\langle h_n, h_{n-kN} \rangle_n = \delta_k. \tag{4.16}
\]
DEFINITION 4.3 (Bandlimited subspace of $\ell^2(\mathbb{Z})$) For any $\Omega \in (0, 2\pi)$, a sequence $x \in \ell^2(\mathbb{Z})$ is said to have bandwidth $\Omega$ when its discrete-time Fourier transform $X(e^{j\omega})$ satisfies

$$X(e^{j\omega}) = 0 \quad \text{for all } \omega \in [-\pi, -\Omega/2) \cup (\Omega/2, \pi]. \quad (4.17)$$

All $x \in \ell^2(\mathbb{Z})$ with bandwidth $\Omega$ form a subspace that is denoted $BL[-\Omega/2, \Omega/2]$.

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**Figure 4.9:** Sampling followed by interpolation.

PROPOSITION 4.4 (Discrete-time sampling) Let $h$ be the ideal lowpass filter with cut-off frequency $\pi/N$ and gain $N$ as given in (4.19), and let $\hat{h}$ be the ideal lowpass filter with the same cut-off frequency and gain 1 as given in (4.20). Then the sampling and interpolation system shown in Figure 4.9 has the following properties:

1. If $x \in BL[-\Omega/2, \Omega/2]$ and $\Omega < 2\pi/N$, then $\hat{x} = x$.
2. For an arbitrary $x \in \ell^2(\mathbb{Z})$, the system computes the orthogonal projection of $x$ onto $BL[-\pi/N, \pi/N]$.

$$H(e^{j\omega}) = \begin{cases} N, & \text{for } |\omega| \leq \pi/N; \\ 0, & \text{otherwise.} \end{cases} \quad (4.19a)$$

$$h_n = \text{sinc}(\pi n/N). \quad (4.19b)$$

$$\hat{h}_n = \frac{1}{N} \text{sinc}(\pi n/N) \quad \text{DTFT} \quad \tilde{H}(e^{j\omega}) = \begin{cases} 1, & \text{for } |\omega| \leq \pi/N; \\ 0, & \text{otherwise} \end{cases} \quad (4.20)$$
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Figure 4.16: Sampling and interpolation for continuous-time functions. (a) Sampling is filtering followed by function evaluation at integer multiples of a sampling period $T$, $y_n = (h \ast x)(nT)$. (b) Interpolation is conversion to a Dirac delta function train with spacing $T$ followed by filtering, $\hat{x}(t) = \sum_k y_k h(t - kT)$.

DEFINITION 4.5 (SHIFT-INvariant SUBSPACE OF $L^2(\mathbb{R})$) A subspace $W \subset L^2(\mathbb{R})$ is a shift-invariant subspace (SISS) with respect to shift $\tau \in \mathbb{R}^+$ when $x(t) \in W$ implies $x(t - k\tau) \in W$ for every integer $k$. In addition, $w \in L^2(\mathbb{R})$ is called a generator of $W$ when $W = \text{span}(\{w(t - k\tau)\}_{k \in \mathbb{Z}})$.

Figure 4.18: Sampling and interpolation where the sampling prefilter and interpolation postfilter are related by time reversal.

PROPOSITION 4.6 (SAMPLING FOR SISS OF $L^2(\mathbb{R})$) Consider the system shown in Figure 4.18 in which the sampling prefilter is the time-reversed version of the interpolation postfilter. Let $S$ denote the SISS with respect to shift $T$ generated by $h$. If the filter satisfies (4.33), then:

1. Any input $x \in S$ is reproduced perfectly, that is, $\hat{x} = x$.
2. Any input $x \not\in S$ is approximated optimally in the least-squares sense, that is $\|\hat{x} - x\|_2$ is minimized among $\hat{x} \in S$.

\[ \langle h(t), h(t - nT) \rangle_t = \delta_n. \] (4.33)
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Definition 4.7 (Bandlimited subspace of $L^2(\mathbb{R})$) For any $\Omega \in \mathbb{R}^+$, a function $x \in L^2(\mathbb{R})$ is said to have bandwidth $\Omega$ when its Fourier transform $X(\omega)$ satisfies

$$X(\omega) = 0 \quad \text{for all } \omega \notin [-\Omega/2, \Omega/2].$$

(4.34)

All $x \in L^2(\mathbb{R})$ with bandwidth $\Omega$ form a subspace that is denoted $BL[-\Omega/2, \Omega/2]$.

Figure 4.19: Sampling and interpolation with no sampling prefilter. This is analyzed to show that $x \in BL[-\Omega/2, \Omega/2]$ implies $\hat{x} = x$ when $\Omega < 2\pi/T$.

Theorem 4.8 (Sampling Theorem (Shannon-Whittaker-Kotelnikov-Raabe)) Let $h$ be the ideal lowpass filter with cut-off frequency $\pi/T$ and gain $T$ as given in (4.37), and let the input to the sampling and interpolation system shown in Figure 4.19 satisfy $x \in BL[-\Omega/2, \Omega/2]$. If $\Omega < 2\pi/T$, then $\hat{x} = x$. In particular, this means that the samples $\{x(nT)\}_{n \in \mathbb{Z}}$ are a sufficient characterization of $x(t)$ and the reconstruction formula

$$x(t) = \sum_{n \in \mathbb{Z}} x(nT) \text{sinc}(\pi t/T - n)$$

holds.

$$H(\omega) = \begin{cases} T, & \text{for } |\omega| \leq \pi/T; \\ 0, & \text{otherwise.} \end{cases}$$

(4.37a)

$$h(t) = \text{sinc}(\pi t/T).$$

(4.37b)
Beyond shift-invariant sampling

We have sampling for [notation of Unser (2000)]

\[ V(\varphi) = \left\{ f(t) = \sum_{k \in \mathbb{Z}} c_k \varphi(t - k) : c \in \ell^2(\mathbb{Z}) \right\} \]

Alternative:
- Recover \( f \in V(\varphi) \) from non-uniform samples
- More complicated but remains linear

Non-subspace sampling [Vetterli, Marziliano, and Blu (2002)]:

\[ M(\varphi) = \left\{ f(t) = \sum_{k \in \mathbb{Z}} c_k \varphi(t - t_k) : c \in \ell^2(\mathbb{Z}), t_k \text{ distinct} \right\} \]
Exponential fitting

\[ x(t) = \sum_{k=0}^{K-1} c_k (u_k)^t \]
\[ x_n = \sum_{k=0}^{K-1} c_k (u_k)^n \]

c_k and u_k unknown

Fitting exponentials to data is an old problem
- u_k on unit circle ⇒ estimating line spectra
- u_k not on unit circle is the “original” problem in chemistry, physics, … (e.g., Prony (1795))

Reconstruction of semilinear signal reduced to exponential fitting

Exponential fitting

\[ x_n = \sum_{k=0}^{K-1} c_k (u_k)^n, \quad n = 0, 1, \ldots, N - 1 \]

\[ X(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{K-1} c_k (u_k)^n z^{-n} = \sum_{k=0}^{K-1} c_k \frac{1}{1 - u_k z^{-1}} \]

\[ h_n * x_n = 0, \quad n = K + 1, \ldots, N - 1, \quad H(z) = \prod_{k=0}^{K-1} (1 - u_k z^{-1}) \]

\[ \begin{bmatrix} x_{K-1} & x_{K-2} & \cdots & x_0 \\ x_K & x_{K-1} & \cdots & x_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_{2K-2} & x_{2K-3} & \cdots & x_{K-1} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_K \end{bmatrix} = - \begin{bmatrix} x_K \\ x_{K+1} \\ \vdots \\ x_{2K-1} \end{bmatrix} \]

- \(H(z)\) called the annihilating filter
- Need \(2K\) samples \(\{x_n\}_{n=0}^{2K-1}\) to solve for \(K\) unknowns \(\{h_n\}_{n=1}^{K}\)
- \(\{u_k\}_{k=1}^{K}\) are roots of polynomial \(H(z)\)