
Today:
- Fitting sum of exponentials
- Sampling based on finite rate of innovation
- Fundamentals of scalar quantization
- Fundamentals of transform coding

Next time:
- Time-frequency localization (Chapter 6)
- [Omitting Sections 5.2-5.3]
- [Sections 5.4-5.6 deferred to later in semester, except as covered today]

Reading:
- Sections 4.6-4.7
- Portions of some of my old writings will be placed on Stellar

### Exponential fitting

\[
x(t) = \sum_{k=0}^{K-1} c_k (u_k)^t \\
x_n = \sum_{k=0}^{K-1} c_k (u_k)^n
\]

Fitting exponentials to data is an *old* problem
- \(u_k\) on unit circle \(\Rightarrow\) estimating line spectra
- \(u_k\) not on unit circle is the “original” problem in chemistry, physics, … (e.g., Prony (1795))

Reconstruction of semilinear signal reduced to exponential fitting
Exponential fitting

\[ x_n = \sum_{n=0}^{N-1} c_k(u_k)^n, \quad n = 0, 1, \ldots, N - 1 \]

\[ X(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{K-1} c_k(u_k)^n z^{-n} = \sum_{k=0}^{K-1} c_k z^{-u_k-1} \]

\[ h_n * x_n = 0, \quad n = K + 1, \ldots, N - 1, \quad H(z) = \prod_{k=0}^{K-1} (1 - u_k z^{-1}) \]

- \( H(z) \) called the annihilating filter
- Need \( 2K \) samples \( \{x_n\}_{n=0}^{2K} \) to solve for \( K \) unknowns \( \{h_n\}_{n=1}^{K} \)
- \( \{u_k\}_{k=0}^{K-1} \) are roots of polynomial \( H(z) \)

Basics of scalar quantization

OED: To approximate (a signal varying continuously in amplitude) by one whose amplitude is restricted to a prescribed set of discrete values.

Quantizers are most often uniform:
High-resolution analysis of uniform quantizer

\[ D = \mathbb{E}[(X - q(X))^2] \]

\[ \approx \sum_{i=1}^{N} \int_{y_i - \Delta/2}^{y_i + \Delta/2} (x - y_i)^2 f(x) \, dx \quad \text{neglecting overload} \]

\[ \approx \sum_{i=1}^{N} f(y_i) \int_{y_i - \Delta/2}^{y_i + \Delta/2} (x - y_i)^2 \, dx \quad \text{by continuity} \]

\[ = \frac{\Delta^2}{12} \sum_{i=1}^{N} f(y_i) \Delta \]

\[ \approx \frac{\Delta^2}{12} \int_{y_{N+1}}^{y_{N+1} + \Delta/2} f(x) \, dx \]

\[ \approx \frac{\Delta^2}{12} \quad \text{neglecting overload} \]

If \( \text{support}(f(x)) \subset \text{limiting granular region} \),

\[ \lim_{\Delta \to 0} \frac{D}{\Delta^2/12} = 1 \]

High-res. analysis of general scalar quantizer

Consider \( K \)-cell quantization of \( X \in \mathbb{R} \) with smooth pdf \( f_X(x) \)

- optimal quantizer: regular, similarly-sized neighboring cells
- assume overload distortion is negligible
- quantizer described by (normalized) point density \( \lambda(x) \)

\( \Delta \lambda(x) \): approx. fraction of points in \( [x - \frac{1}{2} \Delta, x + \frac{1}{2} \Delta] \)
High-res. analysis of general scalar quantizer

- express MSE using $\lambda$:
  \[ D = \frac{1}{12K^2} \int \frac{f_X(x)}{\lambda^2(x)} \, dx \, o(1/K^2) \]

Fixed-rate quantization (minimize MSE given $K$)
- optimally, $\lambda(x) \sim f_X^{1/3}(x)$

Optimization of point density (fixed rate)

Minimizing $D \approx \frac{1}{12} \frac{1}{N^2} \int \frac{f(x)}{\lambda^2(x)} \, dx$:

Hölder’s inequality: When $\frac{1}{a} + \frac{1}{b} = 1$,

\[ \int u(x)v(x) \, dx \leq (u(x)^a \, dx)^{1/a} (v(x)^b \, dx)^{1/b}, \]

with equality if and only if $u(x)^a = c \cdot v(x)^b$. 
Optimization of point density (fixed rate)

Let \( u(x) = \left( \frac{f(x)}{\lambda^2(x)} \right)^{1/3} \), \( v(x) = \lambda^{2/3}(x) \), \( \alpha = 3 \), \( b = 3/2 \). Then

\[
\int \left( \frac{f(x)}{\lambda^2(x)} \right)^{1/3} \lambda^{2/3}(x) \, dx \leq \left( \int \frac{f(x)}{\lambda^2(x)} \, dx \right)^{1/3} \left( \int \lambda(x) \, dx \right)^{2/3}
\]

Distortion minimized by \( \lambda(x) = \frac{f^{1/3}(x)}{\int f^{1/3}(x') \, dx'} \)

\[
\delta(R) \approx \frac{1}{12} \left( \int f^{1/3}(x) \, dx \right)^3 2^{2-2R}
\]

for Gaussian

High-res. quantization: General variable-rate

With high-resolution approximations, optimal point density is constant.

Uniform quantization gives

\[
H(q(X)) \approx h(X) - \log \Delta,
\]

where \( h(X) = - \int f(x) \log f(x) \, dx \)

\[
\Rightarrow \Delta \approx 2^{h(X)} 2^{-R}
\]

\[
\Rightarrow \delta(R) \approx \frac{1}{12} \frac{2^{2h(X)}}{2\pi e\sigma^2} 2^{2-2R}
\]

for Gaussian

Comparing to fixed-rate quantization, only space-filling loss
(no oblongitis or point-density loss)
General terminology of source coding

input $x \in$ alphabet $A \subset \mathbb{R}^k$
lossy encoder $\alpha : A \to \mathcal{I}$
reproduction decoder $\beta : \mathcal{I} \to \hat{A} \subset \mathbb{R}^k$
lossless encoder $\gamma : \mathcal{I} \to \mathcal{F}$

partition $\mathcal{S} = \{S_i = \alpha^{-1}(i) : i \in \mathcal{I}\}$
(reproduction) codebook $\mathcal{C} = \{\beta(i) : i \in \mathcal{I}\}$ of points, codevectors, or reproduction codewords

Performance measures

distortion $d(x, \hat{x}) = \|x - \hat{x}\|^2 = \sum_{i=1}^{k} |x_i - \hat{x}_i|^2$
rate $R(\alpha, \gamma) = E[r(X)] = \frac{1}{k} E[\ell(\gamma(\alpha(X)))$]
average distortion $D(\alpha, \beta) = \frac{1}{k} E[d(X, \beta(\alpha(X)))$]

operational distortion-rate function
$\delta(R) = \inf_{(\alpha, \gamma, \beta) : R(\alpha, \gamma) \leq R} D(\alpha, \beta)$

operational rate-distortion function
$r(D) = \inf_{(\alpha, \gamma, \beta) : D(\alpha, \gamma) \leq D} R(\alpha, \gamma)$

operational Lagrangian (weighted r-d) function
$L(\lambda) = \inf_{(\alpha, \gamma, \beta)} D(\alpha, \beta) + \lambda R(\alpha, \gamma)$
Examples: 2-D uniform source

Effect of dimension

Performance improves with increasing dimension, but so does complexity. Only “space filling loss” is a fundamental limitation of the quantizer dimension. It decreases \textit{slowly}.

Losses for i.i.d. Gaussian source:
**Linear approximation of random sources**

Let \( x \in \mathbb{R}^N \) be jointly Gaussian with \( E[x] = 0 \) and \( E[xx^T] = R_x \).

(Gaussianity is for concreteness. We are using just second-order properties and constraining ourselves to linear approximation.)

How can we best approximate \( x \) with \( K < N \) terms?

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**Basic transform coding structure**

![Diagram](image)

Transform codes have **constrained structure** for lower complexity:

- \( T \) and \( U \) are linear operators (\( N \times N \) matrices)
- Each \((\alpha_k, \beta_k, \gamma_k)\) is a scalar quantizer

Lower complexity as function of \( N \) \( \Rightarrow \) Use much larger \( N \)

Most audio and image compression is with transform codes.
Transform codes are constrained quantizers

Transform code

Transform code

Oblongitis loss here can be reduced by using a variable-length code

Standard transform coding results

[Huang & Schultheiss (1963)]

Let $x$ be jointly Gaussian and assume the $y_i$'s are independent and the $\alpha_i$'s are optimal fixed-rate quantizers.

Then optimally $U = T^{-1}$ and $T$ is orthogonal.
Standard transform coding results

[See, e.g., Gersho & Gray (1992)]

Let $\mathbf{x}$ be jointly Gaussian and assume $T^{-1} = TT^T = U$. Also assume high-rate approximations of fixed- or variable-rate quantizer performance.

Then $T$ is optimally a KLT ($y_i$'s independent).

More general transform coding result

**Thm. 2 (2000):** Let $T$ be orthogonal and $U = T^{-1} = TT^T$. If $D_k = \sigma_k^2 \mu(R_k)$ for each $k$, a KLT is optimal. (The bit allocation is arbitrary.)

Proof 1: Every Jacobi step in diagonalizing $R_x$ decreases $D$.

Proof 2: Let $(R_1, R_2, \ldots, R_N)$ be any bit allocation. We wish to minimize $D = N^{-1} \sum_{k=1}^{N} \sigma_k^2 f(R_k)$ by manipulating the $\sigma_k^2$'s. The vector $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)$ lies in the convex polytope defined by the permutations of $\lambda(R_x)$. The minimum of $D$ must be attained at a corner.