6.342 Lecture 21 — May 2, 2011

Today:
• Frames

Readings:
• Chapter 10 of *Fourier and Wavelet Signal Processing*

**Definition 1.48 (Frame)** The set of vectors \( \tilde{\Phi} = \{ \tilde{\varphi}_k \}_{k \in \mathcal{J}} \subset H \), where \( \mathcal{J} \) is finite or countably infinite, is called a frame for Hilbert space \( H \) when there exist strictly positive constants \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) (called frame bounds) such that, for any \( x \in H \),

\[
\lambda_{\text{min}} \|x\|^2 \leq \sum_{k \in \mathcal{J}} |(x, \tilde{\varphi}_k)|^2 \leq \lambda_{\text{max}} \|x\|^2. \tag{1.129}
\]

**Definition 1.34 (Riesz Basis)** The set of vectors \( \Phi = \{ \varphi_k \}_{k \in \mathcal{K}} \subset H \), where \( \mathcal{K} \) is finite or countably infinite, is called a Riesz basis for Hilbert space \( H \) when

(i) it is a basis for \( H \); and
(ii) there exist strictly positive constants \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) such that, for any \( x \in H \), the expansion of \( x \) with respect to the basis \( \Phi \),

\[
x = \sum_{k \in \mathcal{K}} \alpha_k \varphi_k,
\]

satisfies

\[
\lambda_{\text{min}} \|x\|^2 \leq \sum_{k \in \mathcal{K}} |\alpha_k|^2 \leq \lambda_{\text{max}} \|x\|^2. \tag{1.76}
\]
Operators Associated with Frames

\[ \Phi : \ell^2(\mathcal{J}) \to H, \quad \text{with} \quad \Phi \alpha = \sum_{k \in \mathcal{J}} \alpha_k \varphi_k \]

\[ \Phi^*: H \to \ell^2(\mathcal{J}), \quad \text{with} \quad (\Phi^* x)_k = \langle x, \varphi_k \rangle, \quad k \in \mathcal{J} \]

\[ \lambda_{\min} I \leq \Phi \Phi^* \leq \lambda_{\max} I \]

Definition 1.49 (Tight Frame) The frame \( \Phi = \{\varphi_k\}_{k \in \mathcal{J}} \subset H \), where \( \mathcal{J} \) is finite or countably infinite, is called a tight frame, or a \( \lambda \)-tight frame, for Hilbert space \( H \) when its best frame bounds are equal, \( \lambda_{\min} = \lambda_{\max} = \lambda \).

For a \( \lambda \)-tight frame, (1.132) simplifies to

\[ \Phi \Phi^* = \lambda I. \quad (1.133) \]

Theorem 1.50 (Tight Frame Expansions) Let \( \Phi = \{\varphi_k\}_{k \in \mathcal{J}} \) be a \( 1 \)-tight frame for Hilbert space \( H \). Analysis of any \( x \) in \( H \) gives expansion coefficients in \( \ell^2(\mathcal{J}) \)

\[ \alpha_k = \langle x, \varphi_k \rangle \quad \text{for} \ k \in \mathcal{J}, \quad \text{or}, \quad \alpha = \Phi^* x. \quad (1.136a) \]

Synthesis with these coefficients yields

\[ x = \sum_{k \in \mathcal{J}} \langle x, \varphi_k \rangle \varphi_k \quad (1.137a) \]

\[ = \Phi \alpha = \Phi \Phi^* x. \quad (1.137b) \]
**Definition 1.52 (Dual pair of frames)** The sets of vectors \( \Phi = \{ \varphi_k \}_{k \in J} \subset H \) and \( \tilde{\Phi} = \{ \tilde{\varphi}_k \}_{k \in J} \subset H \), where \( J \) is finite or countably infinite, are called a dual pair of frames for Hilbert space \( H \) when

(i) each is a frame for \( H \); and
(ii) for any \( x \) in \( H \),

\[
x = \sum_{k \in K} \langle x, \tilde{\varphi}_k \rangle \varphi_k \quad \text{(1.141a)}
\]

\[
= \Phi \tilde{\Phi}^* x. \quad \text{(1.141b)}
\]

**Canonical Dual Frame**

\[
\tilde{\Phi} = (\Phi \Phi^*)^{-1} \Phi
\]

\[
\tilde{\varphi}_k = (\Phi \Phi^*)^{-1} \varphi_k, \quad k \in J.
\]

**Figure 10.7:** Frames at a glance. Tight frames with \( \lambda = 1 \) and unit-norm vectors lead to orthonormal bases.
Robustness to additive noise

\[ \hat{x} = F^\top \hat{\gamma} = F^\top (F x + \eta) = \tilde{F}^* (F x + \eta) = \sum_{k=1}^{M} \langle x, \varphi_k \rangle + \eta_k \tilde{\varphi}_k, \]

\[ x - \hat{x} = \sum_{k=1}^{M} \langle x, \varphi_k \rangle \tilde{\varphi}_k - \sum_{k=1}^{M} \langle x, \varphi_k \rangle + \eta_k \tilde{\varphi}_k = - \sum_{k=1}^{M} \eta_k \tilde{\varphi}_k. \]

The expected squared-$\ell_2$ error per component (mean-squared error) is

\[ \text{MSE} = \frac{1}{N} E \| x - \hat{x} \|^2 = \frac{1}{N} E \left\| \sum_{k=1}^{M} \eta_k \tilde{\varphi}_k \right\|^2 = \frac{1}{N} E \left[ \sum_{i=1}^{M} \sum_{k=1}^{M} \eta_i \eta_k \tilde{\varphi}_i^* \tilde{\varphi}_k \right] = \frac{1}{N} \sum_{i=1}^{M} \sum_{k=1}^{M} \delta_{i,k} \sigma^2 \tilde{\varphi}_i^* \tilde{\varphi}_k = \frac{1}{N} \sigma^2 \sum_{k=1}^{M} \| \tilde{\varphi}_k \|^2, \]

Goyal, Kovacevic & Kelner (2001)

\[ \text{MSE} = \frac{1}{N} \sigma^2 \sum_{k=1}^{M} \| \tilde{\varphi}_k \|^2 = N^{-1} \sigma^2 \text{tr}(\tilde{F} \tilde{F}^*) = N^{-1} \sigma^2 \text{tr}((\tilde{F}^* \tilde{F})^{-1}) = N^{-1} \sigma^2 \text{tr}(V \Lambda^{-1} V^*) = N^{-1} \sigma^2 \text{tr}(\Lambda^{-1}), \]

where $\tilde{F}^* F = V \Lambda V^*$ is the spectral decomposition of $\tilde{F}^* F$. With the $\{\lambda_i\}_{i=1}^{N}$ denoting the eigenvalues of $\tilde{F}^* \tilde{F}$, we have

\[ \text{MSE} = \frac{1}{N} \sigma^2 \sum_{i=1}^{N} \frac{1}{\lambda_i} \quad (19) \]

**Property 2.2.** For any frame, the sum of the eigenvalues of $\tilde{F}^* F$ equals the sum of the lengths of the frame vectors. In particular, for a uniform frame the sum of the eigenvalues equals $M$ all vectors of unit norm

**Proof.** Denote the eigenvalues by $\{\lambda_i\}_{i=1}^{N}$. Using elementary properties of the trace and the definition of $F$,

\[ \sum_{i=1}^{N} \lambda_i = \text{tr}(\tilde{F}^* F) = \text{tr}(F F^*) = \sum_{i=1}^{M} \varphi_i^* \varphi_i = \sum_{i=1}^{M} \| \varphi_i \|^2. \]

Goyal, Kovacevic & Kelner (2001)
Robustness to additive noise

**Theorem 3.1.** When encoding with a uniform frame and decoding with linear reconstruction (16), under the noise model (14), the MSE is minimum if and only if the frame is tight.

**Theorem 3.2.** Consider linear reconstruction (16) with noise $\eta$ satisfying (14) and define the mean-squared error (MSE) by $N^{-1}E\|x - \hat{x}\|^2$. For any frame, the MSE is given by (19) and satisfies

$$B^{-1}\sigma^2 \leq \text{MSE} \leq A^{-1}\sigma^2.$$  (20)

For a uniform frame,

$$\frac{N\sigma^2}{M} \leq \text{MSE} \leq A^{-1}\sigma^2.$$  (21)

For a uniform tight frame,

$$\text{MSE} = \frac{N}{M}\sigma^2 = r^{-1}\sigma^2.$$  (22)

Goyal, Kovacevic & Kelner (2001)

Robustness to erasures

**Theorem 4.4.** Consider encoding with a uniform frame and decoding with linear reconstruction (16), under noise model (14). The MSE averaged over all possible erasures of one frame element,

$$\overline{\text{MSE}_1} = \frac{1}{M} \sum_{k=1}^{M} \text{MSE}_{k|1},$$

is minimum if and only if the original frame is tight. Also, a tight frame minimizes the maximum distortion caused by one erasure

$$\max_{k=1,2,\ldots,M} \text{MSE}_{[k|1].}$$

Goyal, Kovacevic & Kelner (2001)
Robustness to bounded noise and quantization

An estimate $\hat{x}$ is called consistent with $\hat{y} = Q(Fx)$ when $Q(F\hat{x}) = \hat{y}$.

The concept of consistency can be used for both deterministic and random quantization.

In a variety of scenarios, consistency reduces MSE from $O(\frac{N}{M})$ to $O((\frac{N}{M})^2)$.

Fig. 3. Illustration of consistent reconstruction. Goyal, Vetterli & Thao (1998)
Robustness to bounded noise and quantization

TABLE I
ALGORITHM FOR CONSISTENT RECONSTRUCTION FROM A QUANTIZED FRAME EXPANSION

1. Form
\[ \overline{F} = \begin{bmatrix} F & F' \\ -F' & F \end{bmatrix} \quad \text{and} \quad \overline{y} = \begin{bmatrix} \frac{1}{2} \Delta + \hat{y} \\ \frac{1}{2} \Delta - \hat{y} \end{bmatrix}. \]

2. Pick an arbitrary cost function \( c \in \mathbb{R}^N \).
3. Use a linear programming method to find \( \hat{x} \) to minimize \( c^T \hat{x} \) subject to \( \overline{F} \hat{x} \leq \overline{y} \).

(Algorithm written for \( Q \) being a quantizer that rounds to nearest multiple of \( \Delta \).)

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Goyal, Vetterli & Thao (1998)

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Fig. 4: Experimental results for reconstruction from quantized frame expansions. Shows \( O(1/r^2) \) MSE for consistent reconstruction and \( O(1/r) \) MSE for classical reconstruction.

1. A sequence of frames corresponding to oversampled A/D conversion, as given by (6). This is the case in which we have proven an \( O(1/r^2) \) SE upper bound.
2. For \( N = 3, 4, \) and 5, Hardin, Sloane, and Smith have numerically found arrangements of up to 130 points on \( N \)-dimensional unit spheres that maximize the minimum Euclidean norm separation [14].
3. Frames generated by randomly choosing points on the unit sphere according to a uniform distribution.

Goyal, Vetterli & Thao (1998)
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Figure 10.4: A 3-channel filter bank with sampling by 2 implementing a general frame expansion.
Figure 10.16: Sampling grids corresponding to the time-frequency tilings of (a) the DWT (points—nonredundant) and (b) the oversampled DWT (squares—redundant).

Figure 10.17: The synthesis part of the filter bank implementing the oversampled DWT. The samplers are omitted at all the inputs into the bank. The analysis part is analogous.

Figure 10.18: The synthesis part of the equivalent filter bank implementing the oversampled DWT with $J = 3$ levels. The analysis part is analogous.
Figure 10.23: The synthesis part of the equivalent 3-channel filter bank implementing the shift-invariant DWT with $J = 3$ levels. The analysis part is analogous and filters are given in (10.92). This is the same scheme as in Figure 10.18 with all the upsamplers removed.

Figure 10.24: Sampling grid corresponding to the time-frequency tiling of the shift-invariant DWT (points—nonredundant, squares—redundant).