Blur & Fourier Analysis

6.815/6.865
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A bunch of slides by Bill Freeman (MIT) & Alyosha Efros (CMU)
Blur

• Diffraction
• Lens aberrations
• Object movement
• Camera shake

• Can we remove blur computationally?
  – Inverse the blur equation
Image processing

- Sharpen images
- Blur distracting background, skin, faces
- Special effects
One man to the rescue

• Filtering, Convolution, and our friend Joseph Fourier
Simplified model of blur

• Blur is the same everywhere in the image
  – Shift invariant

• For generalizations, see e.g.
  – http://www.di.ens.fr/willow/research/deblurring/
Blur example: spherical aberration

• Pixel value: weighted average of local color
Blur as convolution

• Replace each pixel by a linear combination of its neighbors.
  – only depends on relative position of neighbors
• The prescription for the linear combination is called the “convolution kernel”.

\[
\begin{bmatrix}
10 & 5 & 3 \\
4 & 5 & 1 \\
1 & 1 & 7 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 0 & 0 \\
0 & 0.5 & 0 \\
0 & 1 & 0.5 \\
\end{bmatrix} \quad \begin{bmatrix}
7 \\
\end{bmatrix}
\]

Local image data \quad kernel \quad Modified image data (shown at one pixel)
Linear shift-invariant filtering

- Replace each pixel by a linear combination of its neighbors.
  - only depends on relative position of neighbors
- The prescription for the linear combination is called the “convolution kernel”.
  - same kernel for all pixels

| 10 5 3 | 0 0 0 | 7 |
| 4 5 1 | 0 0.5 0 |
| 1 1 7 | 0 1 0.5 |

Local image data  kernel  Modified image data (shown at one pixel)
Example of linear NON-shift invariant transformation?

• e.g. neutral-density graduated filter (darken high y, preserve small y)
  \[ J(x, y) = I(x, y) \times (1 - y/y_{\text{ymax}}) \]

• Formally, what does linear mean?
  – For two scalars a & b and two inputs x & y:
    \[ F(ax + by) = aF(x) + bF(y) \]

• What does shift invariant mean?
  – For a translation T:
    \[ F(T(x)) = T(F(x)) \]
  – If I blur a translated image, I get a translated blurred image
More formally: Convolution

\[ f[m, n] = I \otimes g = \sum_{k,l} I[m-k, n-l]g[k, l] \]
Convolution (warm-up slide)
Convolution (warm-up slide)

original

Pixel offset

coefficient

1.0

Filtered (no change)
Convolution

original

Pixel offset

Coefficient

1.0

?
shift

original

coefficient

Pixel offset

0

1.0

shifted
Convolution

original

Pixel offset
0

coeficient
0.3

?
Blurring

original

Blurred (filter applied in both dimensions).

Pixel offset

0.3 coefficient
Blur examples

impulse

original

coefficient

Pixel offset

filtered

8

0.3

2.4

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Blur examples

impulse

edge

original

filtered

coefficient

Pixel offset

0.3

0

2.4

original

filtered

8

4

8

4
Questions?
Convolution (warm-up slide)
Convolution (no change)
Convolution

original

2.0

0.33

?
(remember blurring)

- **original**
- **Blurred (filter applied in both dimensions)**

<table>
<thead>
<tr>
<th>Pixel offset</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Sharpening

original

Sharpened original

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Sharpening example

original

coefficient

Sharpened
(differences are accentuated; constant areas are left untouched).
Sharpening

before

after
Oriented filters

Gabor filters at different scales and spatial frequencies

top row shows anti-symmetric (or odd) filters, bottom row the symmetric (or even) filters.
Filtered images

Questions?
Studying convolutions

• Convolution is complicated
  – and hard to inverse
• But at least it’s linear
  \[(f+kg) * h = f * h + k (g * h)\]
• We want to find a better expression
  – Let’s study functions whose behavior is simple under convolution
Blurring: convolution

Same shape, just reduced contrast!!!
This is an eigenvector (output is the input multiplied by a constant)
Eigenvectors

• More precisely, the eigenvectors are $e^{i\omega x}$
• aka $\cos \omega x + i \sin \omega x$
Big Motivation for Fourier analysis

• Sine waves are eigenvectors of the convolution operator
Other motivation for Fourier analysis: sampling

• The sampling grid is a periodic structure
  – Fourier is pretty good at handling that
  – We saw that a sine wave has serious problems with sampling

• Sampling is a linear process
  – but not shift-invariant
Sampling Density

- If we’re lucky, sampling density is enough.
Sampling Density

- If we insufficiently sample the signal, it may be mistaken for something simpler during reconstruction (that's aliasing!)
Sampling recap

• Sampling has bad properties when undersampling (aliasing)
• But they can be analyzed more easily for sine waves
Recap: motivation for sine waves

• Blurring sine waves is simple
  – You get the same sine wave, just scaled down
  – The sine functions are the eigenvectors of the convolution operator

• Sampling sine waves is interesting
  – Get another sine wave
  – Not necessarily the same one! (aliasing)

If we represent functions (or images) with a sum of sine waves, convolution and sampling are easy to study
Questions?
Fourier as change of basis

- Shuffle the data to reveal other information
- E.g., take average & difference: matrix

\[
\begin{bmatrix}
0.5 & 1 \\
0.5 & -1
\end{bmatrix}
\]

Signal

Geometric interpretation

After rotation

Pseudo-Fourier
Fourier as change of basis

- Same thing with infinite-dimensional vectors

Signal  
Geometric interpretation  
After rotation  
Pseudo-Fourier
Fourier transform visualization

\[ F[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f[k,l] e^{-\pi i \left( \frac{km}{M} + \frac{ln}{N} \right)} \]
Question?
Fourier as a change of basis

• Discrete Fourier Transform: just a big matrix

http://www.reindeergraphics.com
To get some sense of what basis elements look like, we plot a basis element --- or rather, its real part --- as a function of x,y for some fixed u, v. We get a function that is constant when (ux+vy) is constant. The magnitude of the vector (u, v) gives a frequency, and its direction gives an orientation. The function is a sinusoid with this frequency along the direction, and constant perpendicular to the direction.
Here \( u \) and \( v \) are larger than in the previous slide.
And larger still...
Question?
Other presentations of Fourier

• Start with Fourier series with periodic signal
• Heat equation
  – more or less special case of convolution
  – iterate -> exponential on eigenvalues
Motivations

• Insights & mathematical beauty
• Sampling rate and filtering bandwidth
• Computation bases
  – FFT: faster convolution
  – E.g. finite elements, fast filtering, heat equation, vibration modes
• Optics: wave nature of light & diffraction
Questions?
The Fourier Transform

- Defined for infinite, aperiodic signals
- Derived from the Fourier series by “extending the period of the signal to infinity”
- The Fourier transform is defined as

\[ X(\omega) = \frac{1}{\sqrt{2\pi}} \int x(t)e^{-j\omega t} dt \]

- \(X(\omega)\) is called the spectrum of \(x(t)\)
- It contains the magnitude and phase of each complex exponential of frequency \(\omega\) in \(x(t)\)
The Fourier Transform

• The inverse Fourier transform is defined as

\[ x(t) = \frac{1}{\sqrt{2\pi}} \int X(\omega)e^{j\omega t} \, d\omega \]

• Fourier transform pair

\[ x(t) \rightarrow X(\omega) \]

• \( x(t) \) is called the *spatial domain* representation
• \( X(\omega) \) is called the *frequency domain* representation
Beware of differences

• Different definitions of Fourier transform
• We use
  \[ X(\omega) = \frac{1}{\sqrt{2\pi}} \int x(t)e^{-j\omega t} \, dt \]
• Other people might exclude normalization or include 2\pi in the frequency
• X might take \omega or j\omega as argument
• Physicist use j, mathematicians use i
Phase

- Don’t forget the phase! Fourier transform results in complex numbers

- Can be seen as sum of sines and cosines

- Or modulus/phase
Phase is important!
Phase is important!

**Figure 6.2** (a) The image shown in Figure 1.4; (b) magnitude of the two-dimensional Fourier transform of (a); (c) phase of the Fourier transform of (a); (d) picture whose Fourier transform has magnitude as in (b) and phase equal to zero; (e) picture whose Fourier transform has magnitude equal to 1 and phase as in (c); (f) picture whose Fourier transform has phase as in (c) and magnitude equal to that of the transform of the picture shown in (g).
Questions?
Duality

Up to details (such as factors of $2\pi$ or signs):
if function $a$ is the Fourier transform of $b$, then $b$ is the Fourier transform of $a$
For example, the Fourier transform of a box is a sinc, and the Fourier transform of a sinc is a box.
Duality

Any theorem that involves the primal and Fourier domains is also true when swapping the two domains.

e.g. shift theorem:

Primal

\[ f(x+a) \]

Fourier

\[ e^{-2\pi i a \omega} F(\omega) \]

\[ e^{-2\pi i ax} f(x) \]

\[ F(\omega+a) \]
Duality

Any theorem that involves the primal and Fourier domains is also true when swapping the two domains.

e.g. scaling theorem:

Primal        Fourier

\[ f(ax) \quad \leftrightarrow \quad \frac{1}{a} F(x/a) \]

\[ \frac{1}{a} f(x/a) \quad \leftrightarrow \quad F(\omega a) \]
Convolution/Modulation

A convolution in one domain is a multiplication in the other one

Recall that Fourier bases are eigenvectors of the convolution
Questions?
Low pass

black means 1, white means 0

http://www.reindeergraphics.com
High pass

http://www.reindeergraphics.com
Filtering in Fourier domain
Analysis of our simple filters

original

Pixel offset

Filtered
(no change)

spectrum: $F(\omega) = 1$

(yes, I am now using the definition without $1/\sqrt{2\pi}$)
Analysis of our simple filters

Pixel offset

Constant magnitude, linearly shifted phase

spectrum:

\[ F(\omega) = e^{-2\pi j\omega \delta} \]
Analysis of our simple filters

Pixel offset

0.3

Low-pass filter

spectrum:
\[ F(\omega) = \frac{\sin(\omega)}{\omega} \]
Analysis of our simple filters

original

sharpened

spectrum:
\[ F(\omega)=2-\text{sinc}(\omega) \]

high-pass filter
Questions?
Fast convolution with FFT

• Convolution cost in primal: \( O(K^2N^2) \)
  – where \( K \) is the width of kernel and \( N \) the width of image

• Convolution cost in Fourier: \( O(N^2) \)
  – because it is diagonal (simple multiplication)

• This suggests it might be valuable to perform convolution in the Fourier domain using an FFT
  – But we need to take into account the FFT’s cost
Convolution with FFT

- Perform Fourier transform of image & kernel
- Multiply Fourier coefficients
- Perform inverse Fourier transform of result
Convolution in primal versus FFT

- 2-d FFT: $O(N^2 \log N)$
  - where $N$ is number of pixels on a side
- Convolution: $K^2 N^2$
  - where $K^2$ is number of samples in kernel
- Say $N=2^{10}$, $K^2=100$.
  FFT~ $3 \times 20 \ 2^{20}$, while convolution ~ $100 \ 2^{20}$
- Rule of thumb: if kernel is smaller than 10-20 pixels, FFT is not worth the extra headache
  - But good for larger kernels
Recall: Words of wisdom

• Careful with the FFT: it assumes a cyclic signal
• Oftentimes, the answer you get mostly shows wraparound artifacts
• For convolution: various options such as mirroring the image or duplicating the boundary values
• For analysis: Proper windowing is needed
  – e.g. multiply function by a smooth function that falls off away from the center so that the boundary is zero
Questions?
Sampling and aliasing
In photos too
More on Samples

- In signal processing, the process of mapping a continuous function to a discrete one is called *sampling*.
- The process of mapping a continuous variable to a discrete one is called *quantization*.
- To represent or render an image using a computer, we must both sample and quantize.
  - Now we focus on the effects of sampling and how to fight them.
Sampling in the Frequency Domain

Original signal

Sampling grid

Sampled signal

Convolution

Multiplication

Fourier Transform

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Reconstruction

• If we can extract a copy of the original signal from the frequency domain of the sampled signal, we can reconstruct the original signal!

• But there may be overlap between the copies.
Guaranteeing Proper Reconstruction

- Separate by removing high frequencies from the original signal (low pass pre-filtering)
- Separate by increasing the sampling density
- If we can't separate the copies, we will have overlapping frequency spectrum during reconstruction → aliasing.
Sampling Theorem

• When sampling a signal at discrete intervals, the sampling frequency must be greater than twice the highest frequency of the input signal in order to be able to reconstruct the original perfectly from the sampled version (Shannon, Nyquist, Whittaker, Kotelnikov, Küpfmüller)