Geometric Pattern Matching

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Matching in a Scene
Eigenfaces

\[= 0.02 + 0.03 + 0.01 + \ldots\]
Formalization: Shapes

• Today:
  – A shape is a set $A$ of points in $\mathbb{R}^2$
  – $|A|=n$

• In general, $A$ could consist of segments etc.
Formalization: (Dis)similarity

- Hausdorff distance
  - $DH(A, B) = \max_{a \in A} \min_{b \in B} ||a - b||$
  - $H(A, B) = \max[ DH(A, B), DH(B, A) ]$

- Earth-Mover Distance
  - Minimum cost of a one-to-one matching between $A$ and $B$
  - $EMD(A, B) = \min_{f: A \rightarrow B, \sum_{a \in A} ||a - f(a)||, f \text{ is } 1:1}$

- Bottleneck matching
  - $BM(A, B) = \min_{f: A \rightarrow B, \max_{a \in A} ||a - f(a)||, f \text{ is } 1:1}$
Alignment

• In general, A and B are not aligned
• So, in general, we want
  – $DH_T(A,B) = \min_{t \in T} DH(t(A),B)$, where
    • $T =$ translations
    • $T =$ translations and rotations
    • ...
  – Same for H
Algorithms

- Exact DH: $O(n \log n)$
- Approx. DH: $O(n)$
- Exact DH$_T$: $O(n^4)$
- Approx. DH$_T$: $O(n^2)$
- Approx. H$_T$: $O(n \log n)$
- DH$_{T+R}=0$: $O()$
Computing $H(A,B)$

- Given $A, B$, how fast can we compute $H(A,B)$?
  - We will compute $DH(A,B)$
  - Construct a Voronoi diagram $V$ for $B$
  - Construct a point location structure for $V$
  - For each $a$ in $A$, find its NN in $B$

- Total time: $O(n \log n)$
Approximate Hausdorff

• Assume we just want an algorithm that:
  – If $\text{DH}(A,B) \leq r$, answers YES
  – If $\text{DH}(A,B) \geq (1+\varepsilon)r$, answers NO

• Algorithm:
  – Impose a grid with cell diameter $\varepsilon r$
  – For each $b \in B$, mark all cells within distance $r$ from $b$
  – For each $a \in A$, check if $a$’s cell is marked

• Time: $O(n/\varepsilon^2)$
Decision Problem

• Again, focus on if $DH_T(A,B) \leq r$
• For $a \in A$, define
  $$T(a) = \{ t : \exists b \in B \ | |t(a)-b|| \leq r \}$$
• $DH_T(A,B) \leq r$ iff $\cap_{a \in A} T(a)$ is non-empty
Algorithm

• Compute the arrangement of all disks
• Check for non-emptiness
• Analysis:
  – Number of disks: $n^2$
  – Size of the arrangement: $O(n^4)$
  – Can compute the arrangement in time $O(n^4)$
  – Total time: $O(n^4)$

• Can be improved to $O(n^3 \log n)$
  [Chew, Goodrich, Huttenlocher, Kedem, Kleinberg, Kravets'93]
Running time

• Running time pretty high
• Can we speed it up?
• A \((1+\epsilon)\)-approximate algorithm for the decision version of DH\(_T\):
  – Impose a grid with pixel diameter \(\epsilon r\)
  – Approximate each disk by \(O(1/\epsilon^2)\) cells
  – Each \(T(a)\) is approximated by \(O(n/\epsilon^2)\) cells
  – Need to check intersection of \(n\ T(a)\)'s
  – \(O(n^2/\epsilon^2)\) time
Approximate Algorithm for $H()$

• Reference point: a point $r(A)$ such that if we set $t=r(B)-r(A)$, then
  $$H( t(A), B ) \leq c \ H_T(A,B)$$

• Note that $t$ transforms $r(A)$ into $r(B)$

• This will give us a $c$-approximate algorithm with time $O(n \log n)$
Constructing a Ref Point

• $r(A) = (\min_{a \in A} a_x, \min_{a \in A} a_y)$

• Introduced in [Alt, Behrends, Blomer’91]

• Improved in [Aichholzer, Alt, Rote’94]
Theorem

Theorem: \( r(A) \) is a reference point with \( c=1+\sqrt{2} \)

Proof:

- Consider optimal \( t \), i.e., s.t. \( H(t(A),B)=H_T(A,B) \)
- What is \( |\min_{a \in A} t(a)_x - \min_{b \in B} b_x| \) ?
- It is at most \( H_T(A,B) \)
- Same for \( |\min_{a \in A} t(a)_y - \min_{b \in B} b_y| \)
- Thus \( ||r(t(A))-r(B)|| \leq \sqrt{2} H_T(A,B) \)
- We can translate \( t(A) \) by a vector of length \( \sqrt{2} H_T(A,B) \), so that \( r(A) \) is aligned with \( r(B) \)
Exact matching under T+R

• Check if $DH_{T+R}(A,B)=0$
  – Can modify the algorithm to allow small error
  – Technique used in practice
Algorithm

- Take any pair $a,a' \in A$, let $r = ||a-a'||$
- Find all pairs $b,b' \in B$ such that $||b-b'|| = r$
- For all such pairs
  - Compute $t$ that transforms $(a,a')$ into $(b,b')$
  - Check if $t(A) \subseteq B$
Analysis

- Choosing $a, a'$: constant time
- Enumerating $b, b'$: $O(n^2)$ time
- Checking match: $O(nP)$, where $P$ is the number of pairs $b, b' \in B$ s.t. $||b - b'|| = r$
- The bound for $P$ is ...
- ...deferred to the next episode 😊
The \((1+\sqrt{2})\) Factor is Tight
(for this reference point)