Today

• **Finite Automata (FAs)**
  – Our third machine model, after circuits and decision trees.

• **Designed to:**
  – **Accept** some strings of symbols.
  – **Recognize** a language, which is the set of strings it accepts.

• **FA takes as its input a string of any length.**
  – One machine for all lengths.
  – Circuits and decision trees use a different machine for each length.

• **Today’s topics:**
  – Finite Automata and the languages they recognize
  – Examples
  – Operations on languages
  – Closure of FA languages under various operations
  – Nondeterministic FAs

• **Reading:** Sipser, Section 1.1.

• **Next:** Sections 1.2, 1.3.
Finite Automata and the languages they recognize
Example 1

- An FA diagram, machine M

- Conventions:
  
  - Start state
  - Accept state
  - Transition from a to b on input symbol 1. Allow self-loops
Example 1

- Example computation:
  - Input word w: 1 0 1 1 0 1 1 1 0 1 1 1 0
  - States: a b a b c a b c d d d

- We say that M accepts w, since w leads to d, an accepting state.
In general...

• A FA $M$ accepts a word $w$ if $w$ causes $M$ to follow a path from the start state to an accept state.

• Some terminology and notation:
  – Finite alphabet of symbols, usually called $\Sigma$.
  – In Example 1 (and often), $\Sigma = \{ 0, 1 \}$.
  – String (word) over $\Sigma$: Finite sequence of symbols from $\Sigma$.
  – Length of $w$, $|w|$
  – $\epsilon$, placeholder symbol for the empty string, $|\epsilon| = 0$
  – $\Sigma^*$, the set of all finite strings of symbols in $\Sigma$
  – Concatenation of strings $w$ and $x$, written $w \circ x$ or $wx$.
  – $L(M)$, language recognized by $M$:
    $$ \{ w \mid w \text{ is accepted by } M \} $$
  – What is $L(M)$ for Example 1?
Example 1

• What is \( L(M) \) for Example 1?
• \( \{ w \in \{0,1\}^* \mid w \text{ contains 111 as a substring} \} \)
• Note: Substring refers to consecutive symbols.
Formal Definition of an FA

- An FA is a 5-tuple \(( Q, \Sigma, \delta, q_0, F )\), where:
  - \( Q \) is a finite set of states,
  - \( \Sigma \) is a finite set (alphabet) of input symbols,
  - \( \delta: Q \times \Sigma \rightarrow Q \) is the transition function,
  - \( q_0 \in Q \), is the start state, and
  - \( F \subseteq Q \) is the set of accepting, or final states.

The arguments of \( \delta \) are a state and an alphabet symbol.

The result is a state.
Example 1

- What is the 5-tuple \((Q, \Sigma, \delta, q_0, F)\)?
- \(Q = \{ a, b, c, d \} \)
- \(\Sigma = \{ 0, 1 \} \)
- \(\delta\) is given by the state diagram, or alternatively, by a table:

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<tbody>
<tr>
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<tr>
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<td>b</td>
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<tr>
<td>(c)</td>
<td>c</td>
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<tr>
<td>(d)</td>
<td>d</td>
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- \(q_0 = a\)
- \(F = \{ d \}\)
Formal definition of computation

- Extend the definition of $\delta$ to input strings and states:
  $\delta^* : Q \times \Sigma^* \rightarrow Q$, state and string yield a state
  $\delta^*( q, w ) = \text{state that is reached by starting at } q \text{ and following } w.$

- Defined recursively:
  $\delta^*( q, \varepsilon ) = q$
  $\delta^*( q, w a ) = \delta( \delta^*( q, w ), a )$

- Or iteratively, compute $\delta^*( q, a_1 a_2 \ldots a_k )$ by:
  $s := q$
  for $i = 1$ to $k$ do $s := \delta( s, a_i )$
Formal definition of computation

• String $w$ is **accepted** if $\delta^*(q_0, w) \in F$, that is, $w$ leads from the start state to an accepting state.

• String $w$ is **rejected** if it isn’t accepted.

• A **language** is any set of strings over some alphabet.

• $L(M)$, language recognized by finite automaton $M = \{ w \mid w$ is accepted by $M \}$.

• A language is **regular**, or **FA-recognizable**, if it is recognized by some finite automaton.
Examples of Finite Automata
Example 2

- Design an FA $M$ with $L(M) = \{ w \in \{0,1\}^* \mid w$ contains 101 as a substring $\}$.  

- Failure from state $b$ causes the machine to remain in state $b$. 

\[
\begin{array}{cccc}
& a & b & c \\
0 & \rightarrow & 1 & \rightarrow \\
1 & \rightarrow & 0 & \rightarrow \\
0,1 & \rightarrow & & \rightarrow \\
\end{array}
\]
Example 3

• \( L = \{ w \in \{ 0,1 \}^* \mid w \) doesn’t contain either 00 or 11 as a substring \}. 

• State \( d \) is a **trap state** = a nonaccepting state that you can’t leave.

• Sometimes we’ll omit some arrows; by convention, they go to a trap state.
Example 4

• $L = \{ w \mid \text{all nonempty blocks of 1s in } w \text{ have odd length} \}$.
• E.g., $\varepsilon$, or 100111000011111, or any number of 0s.
• Initial 0s don’t matter, so start with:

Then 1 also leads to an accepting state, but it should be a different one, to “remember” that the string ends in one 1.
Example 4

- \( L = \{ w \mid \text{all nonempty blocks of 1s in } w \text{ have odd length} \} \).

- From b:
  - 0 can return to a, which can represent either \( \varepsilon \), or any string that is OK so far and ends with 0.
  - 1 should go to a new nonaccepting state, meaning “the string ends with two 1s”.

- Note: c isn’t a trap state---we can accept some extensions.
Example 4

• $L = \{ w \mid \text{all nonempty blocks of 1s in } w \text{ have odd length} \}.$

• From $c$:
  - $1$ can lead back to $b$, since future acceptance decisions are the same if the string so far ends with any odd number of 1s.
    • Reinterpret $b$ as meaning “ends with an odd number of 1s”.
    • Reinterpret $c$ as “ends with an even number of 1s”.
  - $0$ means we must reject the current string and all extensions.
Example 4

• $L = \{ w \mid \text{all nonempty blocks of 1s in } w \text{ have odd length } \}$.

• Meanings of states (more precisely):
  a: Either $\varepsilon$, or contains no bad block (even block of 1s followed by 0) so far and ends with 0.
  b: No bad block so far, and ends with odd number of 1s.
  c: No bad block so far, and ends with even number of 1s.
  d: Contains a bad block.
Example 5

- $L = \text{EQ} = \{ w \mid w \text{ contains an equal number of 0s and 1s} \}$.
- No FA recognizes this language.
- Idea (not a proof):
  - Machine must “remember” how many 0s and 1s it has seen, or at least the difference between these numbers.
  - Since these numbers (and the difference) could be anything, there can’t be enough states to keep track.
  - So the machine will sometimes get confused and give a wrong answer.
- We’ll turn this into an actual proof next week.
Language Operations
Language operations

• Operations that can be used to construct languages from other languages.
• Recall: A language is any set of strings.
• Since languages are sets, we can use the usual set operations:
  – Union, $L_1 \cup L_2$
  – Intersection, $L_1 \cap L_2$
  – Complement, $L^c$
  – Set difference, $L_1 - L_2$
• We also have new operations defined especially for sets of strings:
  – Concatenation, $L_1 \circ L_2$ or just $L_1 L_2$
  – Star, $L^*$
Concatenation

- **Definition:**
  \[ L_1 \circ L_2 = \{ x \, y \mid x \in L_1 \text{ and } y \in L_2 \} \]

  - Pick one string from each language and concatenate them.

- **Example:**
  \[ \Sigma = \{ 0, 1 \}, L_1 = \{ 0, 00 \}, L_2 = \{ 01, 001 \} \]
  \[ L_1 \circ L_2 = \{ 001, 0001, 00001 \} \]

- **Notes:**
  \[ |L_1 \circ L_2| \leq |L_1| \times |L_2|, \text{ not necessarily equal.} \]
  \[L \circ L \text{ does not mean } \{ x \, x \mid x \in L \}, \text{ but rather, } \{ x \, y \mid x \text{ and } y \text{ are both in } L \}.\]
Concatenation

• $L_1 \circ L_2 = \{ x \, y \mid x \in L_1 \text{ and } y \in L_2 \}$

• Example:
  \[ \Sigma = \{ 0, 1 \}, \quad L_1 = \{ 0, 00 \}, \quad L_2 = \{ 01, 001 \} \]
  \[ L_1 \circ L_2 = \{ 001, 0001, 00001 \} \]
  \[ L_2 \circ L_2 = \{ 0101, 01001, 00101, 001001 \} \]

• Example: $\emptyset \circ L$
  \[ \{ x \, y \mid x \in \emptyset \text{ and } y \in L \} = \emptyset \]

• Example: $\{ \varepsilon \} \circ L$
  \[ \{ x \, y \mid x \in \{ \varepsilon \} \text{ and } y \in L \} = L \]
Concatenation

- \( L_1 \circ L_2 = \{ x \ y \mid x \in L_1 \text{ and } y \in L_2 \} \)
- Write \( L \circ L \) as \( L^2 \),
  \[ L \circ L \circ \ldots \circ L \text{ as } L^n, \text{ which is } \{ x_1 \ x_2 \ldots \ x_n \mid \text{all } x's \text{ are in } L \} \]
- Example: \( L = \{0, 11\} \)
  \[ L^3 = \{000, 0011, 0110, 01111, 1100, 11011, 11110, 111111\} \]
- Example: \( L = \{0, 00\} \)
  \[ L^3 = \{000, 0000, 00000, 000000\} \]
- Boundary cases:
  \( L^1 = L \)
  Define \( L^0 = \{\varepsilon\} \), for every \( L \).
  - Implies that \( L^0 \ L^n = \{\varepsilon\} \ L^n = L^n \).
  - Special case of general rule \( L^a \ L^b = L^{a+b} \).
The Star Operation

- $L^* = \{ x \mid x = y_1y_2 \ldots y_k \text{ for some } k \geq 0, \text{ where every } y \text{ is in } L \}$
  
  $$= L^0 \cup L^1 \cup L^2 \cup \ldots$$

- Note: $\varepsilon$ is in $L^*$ for every $L$, since it’s in $L^0$.

- Example: What is $\emptyset^*$?
  
  - Apply the definition:
    $$\emptyset^* = \emptyset^0 \cup \emptyset^1 \cup \emptyset^2 \cup \ldots$$

  The rest of these are just $\emptyset$.

  This is $\{ \varepsilon \}$, by the convention that $L^0 = \{ \varepsilon \}$.

  $$= \{ \varepsilon \}.$$
The Star Operation

- \( L^* = L^0 \cup L^1 \cup L^2 \cup \ldots \)

- Example: What is \( \{ a \}^* \)?
  - Apply the definition:
    \[
    \{ a \}^* = \{ a \}^0 \cup \{ a \}^1 \cup \{ a \}^2 \cup \ldots \\
    = \{ \varepsilon \} \cup \{ a \} \cup \{ a a \} \cup \ldots \\
    = \{ \varepsilon, a, a a, a a a, \ldots \}
    \]
  - Abbreviate this to just \( a^* \).
  - Note this is not just one string, but a set of strings---any number of a’s.
The Star Operation

- \( L^* = L^0 \cup L^1 \cup L^2 \cup \ldots \)

- Example: What is \( \Sigma^* \)?
  - We’ve already defined this to be the set of all finite strings over \( \Sigma \).
  - But now it has a new formal definition:
    \[
    \Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots
    \]
    \[
    = \{ \varepsilon \} \cup \{ \text{strings of length 1 over } \Sigma \} \\
    \cup \{ \text{strings of length 2 over } \Sigma \} \\
    \cup \ldots
    \]
    \[
    = \{ \text{all finite strings over } \Sigma \}
    \]
  - Consistent.
Summary: Language Operations

• Set operations: Union, intersection, complement, set difference

• New language operations: Concatenation, star

• Regular operations:
  – Of these six operations, we identify three as regular operations: union, concatenation, star.
  – We’ll revisit these next time, when we define regular expressions.
Closure of regular (FA-recognizable) languages under all six operations
Closure under operations

• The set of FA-recognizable languages is closed under all six operations (union, intersection, complement, set difference, concatenation, star).
• This means: If we start with FA-recognizable languages and apply any of these operations, we get another FA-recognizable language (for a different FA).

• **Theorem 1:** FA-recognizable languages are closed under complement.
• **Proof:**
  - Start with a language $L_1$ over alphabet $\Sigma$, recognized by some FA, $M_1$.
  - Produce another FA, $M_2$, with $L(M_2) = \Sigma^* - L(M_1)$.
  - Just interchange accepting and non-accepting states.
• **Theorem 1:** FA-recognizable languages are closed under complement.
• **Proof:** Interchange accepting and non-accepting states.
• **Example:** FA for \{ w \mid w \text{ does not contain 111} \}
  - Start with FA for \( \{ w \mid w \text{ contains 111} \} \):
Closure under complement

- **Theorem 1:** FA-recognizable languages are closed under complement.
- **Proof:** Interchange accepting and non-accepting states.
- **Example:** FA for \{ w | w does not contain 111 \}
  - Interchange accepting and non-accepting states:
Closure under intersection

• **Theorem 2:** FA-recognizable languages are closed under intersection.

• **Proof:**
  – Start with FAs $M_1$ and $M_2$ for the same alphabet $\Sigma$.
  – Get another FA, $M_3$, with $L(M_3) = L(M_1) \cap L(M_2)$.
  – Idea: Run $M_1$ and $M_2$ “in parallel” on the same input. If both reach accepting states, accept.
  – Example:
    • $L(M_1)$: Contains substring 01.
    • $L(M_2)$: Odd number of 1s.
    • $L(M_3)$: Contains 01 and has an odd number of 1s.
Closure under intersection

• Example:
  M₁: Substring 01
  M₂: Odd number of 1s
  M₃:
Closure under intersection, general rule

• Assume:
  – $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
  – $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$

• Define $M_3 = (Q_3, \Sigma, \delta_3, q_{03}, F_3)$, where
  – $Q_3 = Q_1 \times Q_2$
    • Cartesian product, $\{(q_1,q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2\}$
  – $\delta_3 ((q_1,q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$
  – $q_{03} = (q_{01}, q_{02})$
  – $F_3 = F_1 \times F_2 = \{(q_1,q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2\}$
Closure under union

• **Theorem 3:** FA-recognizable languages are closed under union.

• **Proof:**
  – Similar to intersection.
  – Start with FAs $M_1$ and $M_2$ for the same alphabet $\Sigma$.
  – Get another FA, $M_3$, with $L(M_3) = L(M_1) \cup L(M_2)$.
  – Idea: Run $M_1$ and $M_2$ “in parallel” on the same input. If either reaches an accepting state, accept.
  – Example:
    • $L(M_1)$: Contains substring 01.
    • $L(M_2)$: Odd number of 1s.
    • $L(M_3)$: Contains 01 or has an odd number of 1s.
Closure under union

• Example:
  M₁: Substring 01
  M₂: Odd number of 1s
  M₃:
Closure under union, general rule

• Assume:
  – $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
  – $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$

• Define $M_3 = (Q_3, \Sigma, \delta_3, q_{03}, F_3)$, where
  – $Q_3 = Q_1 \times Q_2$
    • Cartesian product, ${(q_1,q_2) | q_1 \in Q_1 \text{ and } q_2 \in Q_2}$
  – $\delta_3 ((q_1,q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$
  – $q_{03} = (q_{01}, q_{02})$
  – $F_3 = \{ (q_1,q_2) | q_1 \in F_1 \text{ or } q_2 \in F_2 \}$
Closure under set difference

• **Theorem 4**: FA-recognizable languages are closed under set difference.

• **Proof**:
  – Similar proof to those for union and intersection.
  – Alternatively, since $L_1 - L_2$ is the same as $L_1 \cap (L_2)^c$, we can just apply Theorems 2 and 3.
Closure under concatenation

• **Theorem 5:** FA-recognizable languages are closed under concatenation.

• **Proof:**
  - Start with FAs $M_1$ and $M_2$ for the same alphabet $\Sigma$.
  - Get another FA, $M_3$, with $L(M_3) = L(M_1) \circ L(M_2)$, which is \( \{ x_1 x_2 \mid x_1 \in L(M_1) \text{ and } x_2 \in L(M_2) \} \)
  - Idea: ???
    • Attach accepting states of $M_1$ somehow to the start state of $M_2$.
    • But we have to be careful, since we don’t know when we’re done with the part of the string in $L(M_1)$---the string could go through accepting states of $M_1$ several times.
Closure under concatenation

- **Theorem 5:** FA-recognizable languages are closed under concatenation.

- **Example:**
  - $\Sigma = \{0, 1\}$, $L_1 = \Sigma^*$, $L_2 = \{0\} \{0\}^*$ (just 0s, at least one).
  - $L_1 L_2 = \text{strings that end with a block of at least one 0}$
  - $M_1$:
  - $M_2$:

- How to combine?
- We seem to need to “guess” when to shift to $M_2$.
- Leads to our next model, NFAs, which are FAs that can guess.
Closure under star

• **Theorem 6:** FA-recognizable languages are closed under star.

• **Proof:**
  – Start with FA $M_1$.
  – Get another FA, $M_2$, with $L(M_2) = L(M_1)^*$.  
  – Same problems as for concatenation---need guessing.
  – …
  – We’ll define NFAs next, then return to complete the proofs of Theorems 5 and 6.
Nondeterministic Finite Automata
Nondeterministic Finite Automata

- Generalize FAs by adding **nondeterminism**, allowing several alternative computations on the same input string.
- Ordinary deterministic FAs follow one path on each input.
- Two changes:
  - Allow $\delta(q, a)$ to specify more than one successor state:
    - ![Diagram](image1)
  - Add $\epsilon$-transitions, transitions made “for free”, without “consuming” any input symbols.
    - ![Diagram](image2)
- Formally, combine these changes:
Formal Definition of an NFA

• An NFA is a 5-tuple \(( Q, \Sigma, \delta, q_0, F )\), where:
  – \( Q \) is a finite set of states,
  – \( \Sigma \) is a finite set (alphabet) of input symbols,
  – \( \delta: Q \times \Sigma_{\epsilon} \rightarrow P(Q) \) is the transition function,
  – \( q_0 \in Q \), is the start state, and
  – \( F \subseteq Q \) is the set of accepting, or final states.

The arguments are a state and either an alphabet symbol or \( \epsilon \). \( \Sigma_{\epsilon} \) means \( \Sigma \cup \{ \epsilon \} \).

The result is a set of states.
Formal Definition of an NFA

• An NFA is a 5-tuple \(( Q, \Sigma, \delta, q_0, F )\), where:
  – \( Q \) is a finite set of states,
  – \( \Sigma \) is a finite set (alphabet) of input symbols,
  – \( \delta : Q \times \Sigma \rightarrow P(Q) \) is the transition function,
  – \( q_0 \in Q \), is the start state, and
  – \( F \subseteq Q \) is the set of accepting, or final states.

• How many states in \( P(Q) \)?
  \( 2^{\lvert Q \rvert} \)

• Example: \( Q = \{ a, b, c \} \)
  \( P(Q) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \} \)
NFA Example 1

Q = \{ a, b, c \}
Σ = \{ 0, 1 \}
q₀ = a
F = \{ c \}

δ:

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NFA Example 2

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Next time…

• NFAs and how they compute
• NFAs vs. FAs
• Closure of regular languages under languages operations, revisited
• Regular expressions
• Regular expressions denote FA-recognizable languages.

• Reading: Sipser, Sections 1.2, 1.3