Topics for Counting

Disclaimer: This checklist includes topics that we have covered on counting. It is meant to assist students in studying for Midterm 3, but in no way is this checklist guaranteed to be comprehensive, nor is it a substitute for reviewing official course materials (e.g., lecture videos/slides, the textbook, class problems, problem sets...). The length of the sub-lists or any wording in this document should not be interpreted as suggestive regarding what problems will be on the actual Midterm 3. This checklist is a new resource still in development; please help improve it by sending corrections and suggestions to yingz@mit.edu.

Counting Basics

- **Bijection Rule**
  - Use: counting set $A$ by counting set $B$, which is easier to count
  - Condition: there is a bijection between $A$ and $B$
  - By-conclusion of the bijection: $|A| = |B|$ without calculating either $|A|$ or $|B|$.
  - Signature example: $A :=$ all ways to select 12 donuts from 5 varieties
    $B :=$ all 16-bit sequences with 12 zeros and 4 ones

- **Product Rule**
  - If $P_1, P_2, \ldots, P_n$ are finite sets, then $|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdots |P_n|$
  - Bijecting subsets of an $n$-element set to length-$n$ bit strings:
    $|\{0, 1\}^n| = |\{0, 1\}_1 \times \{0, 1\}_2 \times \cdots \times \{0, 1\}_n| = |\{0, 1\}|^n = 2^n$
  - Signature examples: $P_1 \times P_2 \times \cdots \times P_n :=$ different outfit/different daily diet/…

- **Sum Rule**
  - If $A_1, A_2, \ldots, A_n$ are disjoint sets, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$

- **Counting using Sum Rule & Product Rule & Bijection**

- **Generalized Product Rule**
  - Let $S$ be a set of length-$k$ sequence, where the $i$-th element could take on $n_i$ possible values, $|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$

- **Permutations**
  - $|\text{permutations of an } n\text{-element set}| = n!$
  - $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, by Stirling’s Formula

- **Division Rule**
  - If $f: A \to B$ is $k$-to-$1$, then $|A| = k \cdot |B|$
Use: counting set \( B \) by counting set \( A \), which is easier to count

Signature examples:
- placements of 2 identical rooks in different columns and rows
- King Arthur’s round table

Counting subsets

\( \binom{n}{k} := \text{number of } k\text{-element subsets of an } n\text{-element set} \)  
(a.k.a. \( n \) choose \( k \))

\[
\binom{n}{k} = \frac{n!}{k! (n-k)!}.
\]

\[
\binom{n}{0} = \frac{n!}{0! (n-0)!} = 1
\]

Multinomial coefficient

For \( n, k_1, \ldots, k_m \in \mathbb{N} \) and \( k_1 + k_2 + \cdots + k_m = n \),

\[
\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}.
\]

Subset Split Rule:

\[
\binom{n}{k_1, k_2, \ldots, k_m} = \text{number of } (k_1, k_2, \ldots, k_m)\text{-splits of an } n\text{-element set}
\]

Bookkeeper Rule:

\[
\binom{n}{k_1 + k_2 + \cdots + k_m} = \text{number of sequences with } k_i\text{ occurrences of the distinct element } l_i
\]

Binomial Theorem: For all \( n \in \mathbb{N} \) and \( a, b \in \mathbb{R} \),

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]

Multinomial Theorem: For all \( n \in \mathbb{N} \),

\[
(z_1 + z_2 + \cdots + z_m)^n = \sum_{\substack{k_1, k_2, \ldots, k_m \in \mathbb{N} \\ k_1 + k_2 + \cdots + k_m = n}} \binom{n}{k_1, k_2, \ldots, k_m} z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}
\]

Count Poker Hands

- Hands with a Four-of-a-King
- Hands with a Full House
- Hands with Two Pairs
- Hands with Every Suit

**Pigeonhole Principle**

- Pigeonhole Principle
  - Informal definition: if more pigeons than holes, then \( \geq 2 \) pigeons must be in the same hole.
Formal definition: if $|A| > |B|$, then every total function $f: A \rightarrow B$, there exist two different elements of $A$ that mapped by $f$ to the same element of $B$. ($A$: set of pigeons; $B$: set of holes)

Key to problems: correctly identify the pigeons, the holes, and $f$

Generalized Pigeonhole Principle: if $|A| > k \cdot |B|$, then every total function $f: A \rightarrow B$ maps at least $k + 1$ different elements of $A$ to the same element of $B$.

Same sum subsets

Magician & Assistant with 5 cards

Magician & Assistant with 4 cards

**Inclusion-Exclusion**

Inclusion-Exclusion Rule: $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$

Union of $n$ sets

$|S_1 \cup S_2 \cup \ldots \cup S_n|$

= sum of individual sets' sizes - sizes of all 2 way intersections
+ sizes of all 3 way intersections - sizes of all 4 way intersections
+ sizes of all 5 way intersections - sizes of all 6 way intersections ... 

Computing Euler’s Function

$\phi(n) = n \prod_{i=1}^{m} (1 - \frac{1}{p_i})$, where

prime factorization of $n$ is $p_1^{e_1} \cdots p_m^{e_m}$ for distinct primes $p_i$

Using Inclusion-Exclusion

**Generating Functions for Counting**

Generating functions

transform problems about sequences into problems about algebraic expressions
simplify many counting problems

template: $F(x) = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots$ (finite or infinite)
notation: $[x^n]F(x) ::= f_n$

convergence/divergence does not matter!

a series $\leftrightarrow$ a generating function BY elements of the series $\leftrightarrow$ coefficients $f_0, f_1, f_2, f_3 \cdots$
For any constant $c$, $[x^n](c \cdot F(x)) = c \cdot [x^n]F(x)$

Counting selections from a set using generating functions

Number of ways to select $n$ of the same object = $f_n$ in the $F(x)$ for the object

Counting selections from disjoint sets using generating functions (a.k.a. the Convolution Rule)

Number of ways to select $n$ of different objects = $f_n$ in the $F(x) = \prod_i G_i(x)$, where $G_i(x)$ is the generating function for the $i$th distinct object

Maclaurin’s Theorem: $[x^n] \frac{1}{(1-x)^k} = \binom{n+k-1}{n}$

Binomial Theorem from Convolution Rule: $[x^n](1 + x)^m = \binom{m}{n}$

Partial fraction (math & practice, more math & more practice!)

Generating Functions for Linear Recurrence

Generating function for Fibonacci numbers ($f_0 ::= 0$, $f_1 ::= 1$, $f_n ::= f_{n-1} + f_{n-2}$)

$F(x) ::= f_0 + f_1 x + f_2 x^2 + \cdots f_n x^n + \cdots$

Find the closed form for $F(x)$ via perturbation method: $F(x) = \frac{x}{1-x-x^2}$

Find $f_n$ via partial fraction: $f_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right)$ (a.k.a. Binet’s formula)

$f_n$ is always an integer, grows exponentially, can also be computed by repeated squaring!

Towers of Hanoi

A stack of 64 increasingly large gold disks on a post (largest disk on the bottom)
Monks move all disks to another post, 1 disk at a time, with the help of a third post, s.t.
Only the top disk from one post can be dropped onto another post
A larger disk can never lie above a smaller post
Question? What is the minimum number of steps ($t_n$) required to complete the move?

Tower of Hanoi recurrence: $t_0 = 0$, $t_n = 2t_{n-1} + 1$, for $n \geq 1$

Generating function for $t_n$: $T(x) ::= t_0 + t_1 x + t_2 x^2 + \cdots t_n x^n + \cdots$

Perturbation & partial fraction yield $T(x) = \frac{1}{1-2x} - \frac{1}{1-x}$, so $t_n = [x^n]T(x) = 2^n - 1$

Solving Generalized Linear Recurrences via generating function (GF)

Generalized linear recurrence = linear recurrence + inhomogeneous term

Combine GF of linear recurrence & GF of the inhomogeneous term
Find coefficients via partial fraction of the combination