Continuing Adventures in NP-Completeness

New! Improved! Now with 20% more knock-your-socks-off NP-C reductions!

NP-completeness...

- A decision problem is in NP if solutions can be verified in polynomial time
- A decision problem $L$ is NP-complete if...
  - $L$ is in NP
  - For all $L_i$ in NP, $L_i \leq_P L$

The Cook-Levin Theorem (1972)

Stephen Cook
Leonid Levin

"Every decision problem in NP can be (quickly) converted (aka "reduced") to a corresponding 3SAT decision problem" (i.e., 3SAT is "NP-complete")

The following problems are NP-complete...

- Vertex Cover (VC) $3SAT \leq_p VC$
- Independent Set (IS) $VC \leq_p IS$
- Clique $IS \leq_p Clique$

Last time...

Stephen Cook, 1982 Turing Award
Richard Karp, 1985 Turing Award

Richard Karp going with the big bowl on 15th of July 2014
Boston, Massachusetts

3SAT $\leq_p$ DHPST

$(x_1 \lor x_2 \lor \overline{x_3}) \land \ldots$

Variables $x_1, \ldots, x_t$ and clauses $C_1, \ldots, C_k$

Directed Hamiltonian Path Problem (aka DHPST)

Theorem:

The Directed Hamiltonian Path Problem is NP-complete.
DHP

- Directed Hamiltonian Path (DHP)
  - Given a directed graph, is there a Hamiltonian path (one that visits every vertex exactly once)
  - Note that we don’t care about the start and end vertex in this case. Otherwise, it’s the same problem as DHPST that we just saw!

- Plan: Show DHPST \( \leq_p \) DHP

UHPST

- Undirected Hamiltonian Path (UHPST) with start and end vertices
  - Given an undirected graph and designated vertices \( s \) and \( t \) is there a Hamiltonian path with \( s \) and \( t \) as endpoints?
  - The lack of edge directionality is a new twist!

- Plan: Show DHPST \( \leq_p \) UHPST

Attempt #1: DHPST \( \leq_p \) UHPST

- DHPST

- UHPST

Hamiltonian Cycle

- Given an undirected graph, does there exist a Hamiltonian Cycle (HC) in the graph?

  UHPST \( \leq_p \) HC

  How about this...

Hamiltonian Cycle

- Given an undirected graph, does there exist a Hamiltonian Cycle (HC) in the graph?

  UHPST \( \leq_p \) HC

  How about this...

Shortest and Longest Paths

- Given an undirected graph \( G \), start vertex \( s \), and destination vertex \( t \)
  - Shortest path: Find the shortest path from \( s \) to \( t \)
  - Longest path: Find the longest path from \( s \) to \( t \)
  - (Paths are assumed to be simple paths!)
Traveling Salesperson Problem (TSP)

Given a completely connected undirected graph with positive edge weights, and a budget B, does there exist a cycle that visits each vertex exactly once and costs at most B?

\[ \text{HC} \leq_p \text{TSP} \]

The "incidence" matrix for the graph

Claim: \( \text{SUBSET SUM is NP-complete.} \)

\[ \{1, 3, 4, 10, 42\}, 47 \]

Proof:

1. \( \text{SUBSET SUM} \in \text{NP} \)

2. \( \text{VC} \leq_p \text{SUBSET SUM} \)

Vertex Cover!!? You're gotta be kidding!