LECTURE 2

LECTURE OUTLINE

• Convex sets and functions
• Epigraphs
• Closed convex functions
• Recognizing convex functions

Reading: Section 1.1
SOME MATH CONVENTIONS

• All of our work is done in $\mathbb{R}^n$: space of $n$-tuples $x = (x_1, \ldots, x_n)$

• All vectors are assumed column vectors

• "'" denotes transpose, so we use $x'$ to denote a row vector

• $x'y$ is the inner product $\sum_{i=1}^{n} x_i y_i$ of vectors $x$ and $y$

• $\|x\| = \sqrt{x'x}$ is the (Euclidean) norm of $x$. We use this norm almost exclusively

• See Appendix A of the textbook for an overview of the linear algebra and real analysis background that we will use. Particularly the following:
  
  – Definition of sup and inf of a set of real numbers
  
  – Convergence of sequences (definitions of lim inf, lim sup of a sequence of real numbers, and definition of lim of a sequence of vectors)

  – Open, closed, and compact sets and their properties

  – Definition and properties of differentiation
A subset $C$ of $\mathbb{R}^n$ is called **convex** if
\[ \alpha x + (1 - \alpha)y \in C, \quad \forall x, y \in C, \forall \alpha \in [0, 1] \]

Operations that preserve convexity
- Intersection, scalar multiplication, vector sum, closure, interior, linear transformations

Special convex sets:
- **Polyhedral sets**: Nonempty sets of the form
  \[ \{ x \mid a'_j x \leq b_j, \ j = 1, \ldots, r \} \]
  (always convex, closed, not always bounded)
- **Cones**: Sets $C$ such that $\lambda x \in C$ for all $\lambda > 0$ and $x \in C$ (not always convex or closed)
• Let $C$ be a convex subset of $\mathbb{R}^n$. A function $f : C \mapsto \mathbb{R}$ is called **convex** if for all $\alpha \in [0, 1]$

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \quad \forall x, y \in C$$

If the inequality is strict whenever $a \in (0, 1)$ and $x \neq y$, then $f$ is called **strictly convex** over $C$.

• If $f$ is a convex function, then all its level sets $\{x \in C \mid f(x) \leq \gamma\}$ and $\{x \in C \mid f(x) < \gamma\}$, where $\gamma$ is a scalar, are convex.
EXTENDED REAL-VALUED FUNCTIONS

- The **epigraph** of a function \( f : X \mapsto [-\infty, \infty] \) is the subset of \( \mathbb{R}^{n+1} \) given by

  \[
  \text{epi}(f) = \{(x, w) \mid x \in X, w \in \mathbb{R}, f(x) \leq w\}
  \]

- The **effective domain** of \( f \) is the set

  \[
  \text{dom}(f) = \{x \in X \mid f(x) < \infty\}
  \]

- We say that \( f \) is **convex** if \( \text{epi}(f) \) is a convex set. If \( f(x) \in \mathbb{R} \) for all \( x \in X \) and \( X \) is convex, the definition “coincides” with the earlier one.

- We say that \( f \) is **closed** if \( \text{epi}(f) \) is a closed set.

- We say that \( f \) is **lower semicontinuous** at a vector \( x \in X \) if \( f(x) \leq \lim \inf_{k \to \infty} f(x_k) \) for every sequence \( \{x_k\} \subset X \) with \( x_k \to x \).
CLOSEDNESS AND SEMICONTINUITY I

- **Proposition**: For a function $f : \mathbb{R}^n \mapsto [-\infty, \infty]$, the following are equivalent:

  (i) $V_\gamma = \{x \mid f(x) \leq \gamma\}$ is closed for all $\gamma \in \mathbb{R}$.
  (ii) $f$ is lower semicontinuous at all $x \in \mathbb{R}^n$.
  (iii) $f$ is closed.

  ![Diagram](image)

- (ii) $\Rightarrow$ (iii): Let $\{(x_k, w_k)\} \subset \text{epi}(f)$ with $(x_k, w_k) \to (\bar{x}, \bar{w})$. Then $f(x_k) \leq w_k$, and

  $$f(\bar{x}) \leq \liminf_{k \to \infty} f(x_k) \leq \bar{w} \quad \text{so} \quad (\bar{x}, \bar{w}) \in \text{epi}(f)$$

- (iii) $\Rightarrow$ (i): Let $\{x_k\} \subset V_\gamma$ and $x_k \to \bar{x}$. Then $(x_k, \gamma) \in \text{epi}(f)$ and $(x_k, \gamma) \to (\bar{x}, \gamma)$, so $(\bar{x}, \gamma) \in \text{epi}(f)$, and $\bar{x} \in V_\gamma$.

- (i) $\Rightarrow$ (ii): If $x_k \to \bar{x}$ and $f(\bar{x}) > \gamma > \liminf_{k \to \infty} f(x_k)$ consider subsequence $\{x_k\}_K \to \bar{x}$ with $f(x_k) \leq \gamma$ - contradicts closedness of $V_\gamma$. 

CLOSEDNESS AND SEMICONTINUITY II

• Lower semicontinuity of a function is a “domain-specific” property, but closeness is not:
  – If we change the domain of the function without changing its epigraph, its lower semicontinuity properties may be affected.
  – **Example**: Define $f : (0, 1) \to [-\infty, \infty]$ and $\hat{f} : [0, 1] \to [-\infty, \infty]$ by
    \[
    f(x) = 0, \quad \forall x \in (0, 1),
    \]
    \[
    \hat{f}(x) = \begin{cases} 
    0 & \text{if } x \in (0, 1), \\
    \infty & \text{if } x = 0 \text{ or } x = 1.
    \end{cases}
    \]

    Then $f$ and $\hat{f}$ have the same epigraph, and both are not closed. But $f$ is lower-semicontinuous at all $x$ of its domain while $\hat{f}$ is not.

• Note that:
  – If $f$ is lower semicontinuous at all $x \in \text{dom}(f)$, it is not necessarily closed
  – If $f$ is closed, $\text{dom}(f)$ is not necessarily closed

• **Proposition**: Let $f : X \mapsto [-\infty, \infty]$ be a function. If $\text{dom}(f)$ is closed and $f$ is lower semicontinuous at all $x \in \text{dom}(f)$, then $f$ is closed.
PROPER AND IMPROPER CONVEX FUNCTIONS

• We say that $f$ is **proper** if $f(x) < \infty$ for at least one $x \in X$ and $f(x) > -\infty$ for all $x \in X$, and we will call $f$ **improper** if it is not proper.

• Note that $f$ is proper if and only if its epigraph is nonempty and does not contain a “vertical line.”

• An improper **closed** convex function is very peculiar: it takes an infinite value ($\infty$ or $-\infty$) at every point.
RECOGNIZING CONVEX FUNCTIONS

- Some important classes of elementary convex functions: Affine functions, positive semidefinite quadratic functions, norm functions, etc.

- **Proposition:** (a) The function \( g : \mathbb{R}^n \mapsto (-\infty, \infty] \) given by

  \[
g(x) = \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x), \quad \lambda_i > 0
  \]

  is convex (or closed) if \( f_1, \ldots, f_m \) are convex (respectively, closed).

(b) The function \( g : \mathbb{R}^n \mapsto (-\infty, \infty] \) given by

\[
g(x) = f(Ax)
\]

where \( A \) is an \( m \times n \) matrix is convex (or closed) if \( f \) is convex (respectively, closed).

(c) Consider \( f_i : \mathbb{R}^n \mapsto (-\infty, \infty], \ i \in I, \) where \( I \) is any index set. The function \( g : \mathbb{R}^n \mapsto (-\infty, \infty] \) given by

\[
g(x) = \sup_{i \in I} f_i(x)
\]

is convex (or closed) if the \( f_i \) are convex (respectively, closed).