LECTURE 4

LECTURE OUTLINE

• Relative interior and closure
• Algebra of relative interiors and closures
• Directions of recession

Reading: Section 1.3.1 and Section 1.4 up to (but not including) Section 1.4.1

Two key facts about convex sets:

• A convex set has nonempty interior (when viewed relative to its affine hull)

• A convex set has nice behavior “at $\infty$”: If a closed convex set contains a half line that starts at one of its points, it contains every translation that starts at another one of its points
RELATIVE INTERIOR

- $x$ is a relative interior point of $C$, if $x$ is an interior point of $C$ relative to $\text{aff}(C)$.

- $\text{ri}(C)$ denotes the relative interior of $C$, i.e., the set of all relative interior points of $C$.

- **Line Segment Principle**: If $C$ is a convex set, $x \in \text{ri}(C)$ and $\overline{x} \in \text{cl}(C)$, then all points on the line segment connecting $x$ and $\overline{x}$, except possibly $\overline{x}$, belong to $\text{ri}(C)$.

- **Proof of case where $\overline{x} \in C$**: See the figure.

- **Proof of case where $\overline{x} \notin C$**: Take sequence $\{x_k\} \subset C$ with $x_k \to \overline{x}$. Argue as in the figure.
ADDITIONAL MAJOR RESULTS

• Let $C$ be a nonempty convex set.
   
   (a) $\text{ri}(C)$ is a nonempty convex set, and has the same affine hull as $C$. 
   
   (b) Prolongation Lemma: $x \in \text{ri}(C)$ if and only if every line segment in $C$ having $x$ as one endpoint can be prolonged beyond $x$ without leaving $C$.

Proof: (a) Assume $0 \in C$. Choose $m$ linearly independent vectors $z_1, \ldots, z_m \in C$, where $m = \text{dimension}(\text{aff}(C))$. Prove that $X \subset \text{ri}(C)$, where

$$X = \left\{ \sum_{i=1}^{m} \alpha_i z_i \mid \sum_{i=1}^{m} \alpha_i < 1, \alpha_i > 0, i = 1, \ldots, m \right\}$$

(b) $\Rightarrow$ is clear by the def. of rel. interior. Reverse: take any $\bar{x} \in \text{ri}(C)$; use Line Segment Principle.
OPTIMIZATION APPLICATION

• A concave function \( f : \mathbb{R}^n \mapsto \mathbb{R} \) that attains its minimum over a convex set \( X \) at an \( x^* \in \text{ri}(X) \) must be constant over \( X \).

Proof: (By contradiction) Let \( x \in X \) be such that \( f(x) > f(x^*) \). Prolong beyond \( x^* \) the line segment \( x \)-to-\( x^* \) to a point \( \bar{x} \in X \). By concavity of \( f \), we have for some \( \alpha \in (0, 1) \)

\[
f(x^*) \geq \alpha f(x) + (1 - \alpha) f(\bar{x}),
\]

and since \( f(x) > f(x^*) \), we must have \( f(x^*) > f(\bar{x}) \) - a contradiction. \( \text{Q.E.D.} \)

• Corollary: A linear function \( f(x) = c'x, \ c \neq 0 \), cannot attain a minimum at an interior point of a convex set.
CALCULUS OF REL. INTERIORS: SUMMARY

- The ri($C$) and cl($C$) of a convex set $C$ “differ very little.”
  - $\text{ri}(C) = \text{ri}(\text{cl}(C))$, $\text{cl}(C) = \text{cl}(\text{ri}(C))$
  - Any point in $\text{cl}(C)$ can be approximated arbitrarily closely by a point in $\text{ri}(C)$.

- Relative interior and closure commute with Cartesian product.

- Relative interior commutes with image under a linear transformation and vector sum, but closure does not.

- Neither relative interior nor closure commute with set intersection.

- “Good” operations: Cartesian product for both, and image for relative interior.

- “Bad” operations: Set intersection for both, and image for closure (need additional assumptions for equality).
CLOSURE VS RELATIVE INTERIOR

• Proposition:

(a) We have $\text{cl}(C) = \text{cl}(\text{ri}(C))$ and $\text{ri}(C) = \text{ri}(\text{cl}(C))$.

(b) Let $\overline{C}$ be another nonempty convex set. Then the following three conditions are equivalent:

(i) $C$ and $\overline{C}$ have the same rel. interior.

(ii) $C$ and $\overline{C}$ have the same closure.

(iii) $\text{ri}(C) \subset \overline{C} \subset \text{cl}(C)$.

Proof: (a) Since $\text{ri}(C) \subset C$, we have $\text{cl}(\text{ri}(C)) \subset \text{cl}(C)$. Conversely, let $\overline{x} \in \text{cl}(C)$. Let $x \in \text{ri}(C)$. By the Line Segment Principle, we have

$$\alpha x + (1 - \alpha)\overline{x} \in \text{ri}(C), \quad \forall \alpha \in (0, 1].$$

Thus, $\overline{x}$ is the limit of a sequence that lies in $\text{ri}(C)$, so $\overline{x} \in \text{cl}(\text{ri}(C))$.

The proof of $\text{ri}(C) = \text{ri}(\text{cl}(C))$ is similar.
LINEAR TRANSFORMATIONS

- Let $C$ be a nonempty convex subset of $\mathbb{R}^n$ and let $A$ be an $m \times n$ matrix.

  (a) We have $A \cdot \text{ri}(C) = \text{ri}(A \cdot C)$.

  (b) We have $A \cdot \text{cl}(C) \subset \text{cl}(A \cdot C)$. Furthermore, if $C$ is bounded, then $A \cdot \text{cl}(C) = \text{cl}(A \cdot C)$.

**Proof:** (a) Intuition: Spheres within $C$ are mapped onto spheres within $A \cdot C$ (relative to the affine hull).

(b) We have $A \cdot \text{cl}(C) \subset \text{cl}(A \cdot C)$, since if a sequence $\{x_k\} \subset C$ converges to some $x \in \text{cl}(C)$ then the sequence $\{Ax_k\}$, which belongs to $A \cdot C$, converges to $Ax$, implying that $Ax \in \text{cl}(A \cdot C)$.

  To show the converse, assuming that $C$ is bounded, choose any $z \in \text{cl}(A \cdot C)$. Then, there exists $\{x_k\} \subset C$ such that $Ax_k \to z$. Since $C$ is bounded, $\{x_k\}$ has a subsequence that converges to some $x \in \text{cl}(C)$, and we must have $Ax = z$. It follows that $z \in A \cdot \text{cl}(C)$. **Q.E.D.**

Note that in general, we may have

$$A \cdot \text{int}(C) \neq \text{int}(A \cdot C), \quad A \cdot \text{cl}(C) \neq \text{cl}(A \cdot C)$$
VECTOR SUMS AND INTERSECTIONS

• Let $C_1$ and $C_2$ be nonempty convex sets.

(a) We have

$$\text{ri}(C_1 + C_2) = \text{ri}(C_1) + \text{ri}(C_2),$$

$$\text{cl}(C_1) + \text{cl}(C_2) \subset \text{cl}(C_1 + C_2)$$

If one of $C_1$ and $C_2$ is bounded, then

$$\text{cl}(C_1) + \text{cl}(C_2) = \text{cl}(C_1 + C_2)$$

(b) We have

$$\text{ri}(C_1 \cap C_2) \subset \text{ri}(C_1 \cap C_2), \quad \text{cl}(C_1 \cap C_2) \subset \text{cl}(C_1) \cap \text{cl}(C_2)$$

If $\text{ri}(C_1) \cap \text{ri}(C_2) \neq \emptyset$, then

$$\text{ri}(C_1 \cap C_2) = \text{ri}(C_1) \cap \text{ri}(C_2), \quad \text{cl}(C_1 \cap C_2) = \text{cl}(C_1) \cap \text{cl}(C_2)$$

**Proof of (a):** $C_1 + C_2$ is the result of the linear transformation $(x_1, x_2) \mapsto x_1 + x_2$.

• Counterexample for (b):

$$C_1 = \{ x \mid x \leq 0 \}, \quad C_2 = \{ x \mid x \geq 0 \}$$

$$C_1 = \{ x \mid x < 0 \}, \quad C_2 = \{ x \mid x > 0 \}$$
RECESSION CONE OF A CONVEX SET

• Given a nonempty convex set $C$, a vector $d$ is a **direction of recession** if starting at any $x$ in $C$ and going indefinitely along $d$, we never cross the relative boundary of $C$ to points outside $C$:

$$x + \alpha d \in C, \quad \forall \ x \in C, \ \forall \ \alpha \geq 0$$

• **Recession cone** of $C$ (denoted by $R_C$): The set of all directions of recession.

• $R_C$ is a cone containing the origin.
RECESSION CONE THEOREM

- Let $C$ be a nonempty closed convex set.
  
  (a) The recession cone $R_C$ is a closed convex cone.

  (b) A vector $d$ belongs to $R_C$ if and only if there exists some vector $x \in C$ such that $x + \alpha d \in C$ for all $\alpha \geq 0$.

  (c) $C$ is compact if and only if $R_C = \{0\}$.

  (d) If $D$ is another closed convex set such that $C \cap D \neq \emptyset$, we have

  $$R_{C \cap D} = R_C \cap R_D$$

  More generally, for any collection of closed convex sets $C_i, i \in I$, where $I$ is an arbitrary index set and $\bigcap_{i \in I} C_i$ is nonempty, we have

  $$R_{\bigcap_{i \in I} C_i} = \bigcap_{i \in I} R_{C_i}$$

- Note an important fact: A nonempty intersection of closed sets $\bigcap_{i \in I} C_i$ is compact if and only if $\bigcap_{i \in I} R_{C_i} = \{0\}$. 
• Let $d \neq 0$ be such that there exists a vector $x \in C$ with $x + \alpha d \in C$ for all $\alpha \geq 0$. We fix $\overline{x} \in C$ and $\alpha > 0$, and we show that $\overline{x} + \alpha d \in C$. By scaling $d$, it is enough to show that $\overline{x} + d \in C$.

For $k = 1, 2, \ldots$, let

$$z_k = x + kd, \quad d_k = \frac{(z_k - \overline{x})}{\|z_k - \overline{x}\|} \|d\|$$

We have

$$\frac{d_k}{\|d\|} = \frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \frac{d}{\|d\|} + \frac{x - \overline{x}}{\|z_k - \overline{x}\|}, \quad \|z_k - x\| \to 1, \quad \frac{x - \overline{x}}{\|z_k - \overline{x}\|} \to 0,$$

so $d_k \to d$ and $\overline{x} + d_k \to \overline{x} + d$. Use the convexity and closedness of $C$ to conclude that $\overline{x} + d \in C$. 

PROOF OF PART (B)
APPLICATION: CLOSURE OF $A \cdot C$

- Let $C$ be a nonempty closed convex, and let $A$ be a matrix with nullspace $N(A)$. Then $AC$ is closed if $R_C \cap N(A) = \{0\}$.

**Proof:** Let $\{y_k\} \subset AC$ with $y_k \to \bar{y}$. Define the nested sequence $C_k = C \cap N_k$, where

$$N_k = \{x \mid Ax \in W_k\}, \quad W_k = \{z \mid \|z - \bar{y}\| \leq \|y_k - \bar{y}\|\}$$

We have $R_{N_k} = N(A)$, $R_{C_k} = R_C \cap N(A) = \{0\}$, so $C_k$ is compact, and $\{C_k\}$ has nonempty intersection. Q.E.D.

![Diagram](image)

- A special case: $C_1 + C_2$ is closed if $C_1$, $C_2$ are closed and one of the two is compact. [Write $C_1 + C_2 = A(C_1 \times C_2)$, where $A(x_1, x_2) = x_1 + x_2$.]

- Related theorem: $A \cdot C$ is closed if $C$ is polyhedral. To be shown later by a more refined method.