LECTURE 6

LECTURE OUTLINE

- Hyperplanes
- Supporting and Separating Hyperplane Theorems
- Strict Separation
- Proper Separation
- Nonvertical Hyperplanes

Reading: Section 1.5
HYPERPLANEs

A hyperplane is a set of the form \( \{x \mid a'x = b\} \), where \( a \) is nonzero vector in \( \mathbb{R}^n \) and \( b \) is a scalar.

We say that two sets \( C_1 \) and \( C_2 \) are separated by a hyperplane \( H = \{x \mid a'x = b\} \) if each lies in a different closed halfspace associated with \( H \), i.e.,

either \( a'x_1 \leq b \leq a'x_2, \forall x_1 \in C_1, \forall x_2 \in C_2, \)

or \( a'x_2 \leq b \leq a'x_1, \forall x_1 \in C_1, \forall x_2 \in C_2 \)

If \( \overline{x} \) belongs to the closure of a set \( C \), a hyperplane that separates \( C \) and the singleton set \( \{\overline{x}\} \) is said be supporting \( C \) at \( \overline{x} \).
• Separating and supporting hyperplanes:

- A separating \( \{ x \mid a' x = b \} \) that is disjoint from \( C_1 \) and \( C_2 \) is called strictly separating:

\[
a' x_1 < b < a' x_2, \quad \forall x_1 \in C_1, \ \forall x_2 \in C_2
\]
SUPPORTING HYPERPLANE THEOREM

• Let $C$ be convex and let $\overline{x}$ be a vector that is not an interior point of $C$. Then, there exists a hyperplane that passes through $\overline{x}$ and contains $C$ in one of its closed halfspaces.

Proof: Take a sequence $\{x_k\}$ that does not belong to $\text{cl}(C)$ and converges to $\overline{x}$. Let $\hat{x}_k$ be the projection of $x_k$ on $\text{cl}(C)$. We have for all $x \in \text{cl}(C)$

$$a'_k x \geq a'_k x_k, \quad \forall \ x \in \text{cl}(C), \forall \ k = 0, 1, \ldots,$$

where $a_k = (\hat{x}_k - x_k)/\|\hat{x}_k - x_k\|$. Let $a$ be a limit point of $\{a_k\}$, and take limit as $k \to \infty$. Q.E.D.
SEPARATING HYPERPLANE THEOREM

- Let $C_1$ and $C_2$ be two nonempty convex subsets of $\mathbb{R}^n$. If $C_1$ and $C_2$ are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that

$$a'x_1 \leq a'x_2, \quad \forall x_1 \in C_1, \forall x_2 \in C_2.$$ 

Proof: Consider the convex set

$$C_1 - C_2 = \{x_2 - x_1 \mid x_1 \in C_1, x_2 \in C_2\}$$

Since $C_1$ and $C_2$ are disjoint, the origin does not belong to $C_1 - C_2$, so by the Supporting Hyperplane Theorem, there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \quad \forall x \in C_1 - C_2,$$

which is equivalent to the desired relation. Q.E.D.
• **Strict Separation Theorem**: Let \( C_1 \) and \( C_2 \) be two disjoint nonempty convex sets. If \( C_1 \) is closed, and \( C_2 \) is compact, there exists a hyperplane that strictly separates them.

![Diagram](image)

**Proof**: (Outline) Consider the set \( C_1 - C_2 \). Since \( C_1 \) is closed and \( C_2 \) is compact, \( C_1 - C_2 \) is closed. Since \( C_1 \cap C_2 = \emptyset \), \( 0 \notin C_1 - C_2 \). Let \( \overline{x}_1 - \overline{x}_2 \) be the projection of 0 onto \( C_1 - C_2 \). The strictly separating hyperplane is constructed as in (b).

• **Note**: Any conditions that guarantee closedness of \( C_1 - C_2 \) guarantee existence of a strictly separating hyperplane. However, there may exist a strictly separating hyperplane without \( C_1 - C_2 \) being closed.
**ADDITIONAL THEOREMS**

- **Fundamental Characterization**: The closure of the convex hull of a set $C \subset \mathbb{R}^n$ is the intersection of the closed halfspaces that contain $C$. (Proof uses the strict separation theorem.)

- We say that a hyperplane properly separates $C_1$ and $C_2$ if it separates $C_1$ and $C_2$ and does not fully contain both $C_1$ and $C_2$.

- **Proper Separation Theorem**: Let $C_1$ and $C_2$ be two nonempty convex subsets of $\mathbb{R}^n$. There exists a hyperplane that properly separates $C_1$ and $C_2$ if and only if

$$\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset$$
PROPER POLYHEDRAL SEPARATION

- Recall that two convex sets $C$ and $P$ such that

$$\text{ri}(C) \cap \text{ri}(P) = \emptyset$$

can be properly separated, i.e., by a hyperplane that does not contain both $C$ and $P$.

- If $P$ is polyhedral and the slightly stronger condition

$$\text{ri}(C) \cap P = \emptyset$$

holds, then the properly separating hyperplane can be chosen so that it does not contain the non-polyhedral set $C$ while it may contain $P$.

On the left, the separating hyperplane can be chosen so that it does not contain $C$. On the right where $P$ is not polyhedral, this is not possible.
**NONVERTICAL HYPERPLANES**

- A hyperplane in $\mathbb{R}^{n+1}$ with normal $(\mu, \beta)$ is nonvertical if $\beta \neq 0$.
- It intersects the $(n+1)$st axis at $\xi = (\mu/\beta)\overline{u} + \overline{w}$, where $(\overline{u}, \overline{w})$ is any vector on the hyperplane.

- A nonvertical hyperplane that contains the epigraph of a function in its “upper” halfspace, provides lower bounds to the function values.
- The epigraph of a proper convex function does not contain a vertical line, so it appears plausible that it is contained in the “upper” halfspace of some nonvertical hyperplane.
• Let $C$ be a nonempty convex subset of $\mathbb{R}^{n+1}$ that contains no vertical lines. Then:

(a) $C$ is contained in a closed halfspace of a non-vertical hyperplane, i.e., there exist $\mu \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ with $\beta \neq 0$, and $\gamma \in \mathbb{R}$ such that $\mu' u + \beta w \geq \gamma$ for all $(u, w) \in C$.

(b) If $(\bar{u}, \bar{w}) \notin \text{cl}(C)$, there exists a nonvertical hyperplane strictly separating $(\bar{u}, \bar{w})$ and $C$.

Proof: Note that $\text{cl}(C)$ contains no vert. line [since $C$ contains no vert. line, $\text{ri}(C)$ contains no vert. line, and $\text{ri}(C)$ and $\text{cl}(C)$ have the same recession cone]. So we just consider the case: $C$ closed.

(a) $C$ is the intersection of the closed halfspaces containing $C$. If all these corresponded to vertical hyperplanes, $C$ would contain a vertical line.

(b) There is a hyperplane strictly separating $(\bar{u}, \bar{w})$ and $C$. If it is nonvertical, we are done, so assume it is vertical. “Add” to this vertical hyperplane a small $\epsilon$-multiple of a nonvertical hyperplane containing $C$ in one of its halfspaces as per (a).