LECTURE 7

LECTURE OUTLINE

• Convex conjugate functions
• Conjugacy theorem
• Support functions and polar cones
• Examples

Reading: Section 1.6
CONJUGATE CONVEX FUNCTIONS

- Consider a function $f$ and its epigraph

Nonvertical hyperplanes supporting $\text{epi}(f)$

$\mapsto$ Crossing points of vertical axis

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x'y - f(x) \}, \quad y \in \mathbb{R}^n.$$ 

- For any $f : \mathbb{R}^n \mapsto [-\infty, \infty]$, its conjugate convex function is defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x'y - f(x) \}, \quad y \in \mathbb{R}^n$$
EXAMPLES

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x'y - f(x) \}, \quad y \in \mathbb{R}^n \]
CONJUGATE OF CONJUGATE

• From the definition

\[ f^{*}(y) = \sup_{x \in \mathbb{R}^n} \{ x' y - f(x) \}, \quad y \in \mathbb{R}^n, \]

note that \( f^{*} \) is convex and closed.

• Reason: \( \text{epi}(f^{*}) \) is the intersection of the epigraphs of the linear functions of \( y \)

\[ x' y - f(x) \]
as \( x \) ranges over \( \mathbb{R}^n \).

• Consider the conjugate of the conjugate:

\[ f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ y' x - f^{*}(y) \}, \quad x \in \mathbb{R}^n. \]

• \( f^{**} \) is convex and closed.

• Important fact/Conjugacy theorem: If \( f \) is closed proper convex, then \( f^{**} = f \).
**CONJUGACY THEOREM - VISUALIZATION**

\[ f^*(y) = \sup_{x \in \mathbb{R}^n} \{ x'y - f(x) \}, \quad y \in \mathbb{R}^n \]

\[ f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ y'x - f^*(y) \}, \quad x \in \mathbb{R}^n \]

- If \( f \) is closed convex proper, then \( f^{**} = f \).
CONJUGACY THEOREM

• Let \( f : \mathbb{R}^n \mapsto (-\infty, \infty] \) be a function, let \( \tilde{\text{cl}} f \) be its convex closure, let \( f^* \) be its convex conjugate, and consider the conjugate of \( f^* \),

\[
f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ y' x - f^*(y) \}, \quad x \in \mathbb{R}^n
\]

(a) We have

\[
f(x) \geq f^{**}(x), \quad \forall x \in \mathbb{R}^n
\]

(b) If \( f \) is closed proper and convex, then

\[
f(x) = f^{**}(x), \quad \forall x \in \mathbb{R}^n
\]

(c) If \( f \) is convex, then properness of any one of \( f, \ f^* \), and \( f^{**} \) implies properness of the other two.

(d) If \( \tilde{\text{cl}} f(x) > -\infty \) for all \( x \in \mathbb{R}^n \), then

\[
\tilde{\text{cl}} f(x) = f^{**}(x), \quad \forall x \in \mathbb{R}^n
\]
PROOF OF CONJUGACY THEOREM (A), (C)

• (a) For all $x, y$, we have $f^*(y) \geq y'x - f(x)$, implying that $f(x) \geq \sup_y \{y'x - f^*(y)\} = f**(x)$.

• (b) By contradiction. Assume there is $(x, \gamma) \in \text{epi}(f**) \setminus \text{epi}(f)$ with $(x, \gamma) \notin \text{epi}(f)$. There exists a non-vertical hyperplane with normal $(y, -1)$ that strictly separates $(x, \gamma)$ and $\text{epi}(f)$. (The vertical component of the normal vector is normalized to -1.)

Consider two parallel hyperplanes, translated to pass through $(x, f(x))$ and $(x, f**(x))$. Their vertical crossing points are $x'y - f(x)$ and $x'y - f**(x)$, and lie strictly above and below the crossing point of the strictly sep. hyperplane. Hence

$$x'y - f(x) > x'y - f**(x)$$

the fact $f \geq f**$. Q.E.D.
A COUNTEREXAMPLE

- A counterexample (with closed convex but improper $f$) showing the need to assume properness in order for $f = f^{**}$:

$$f(x) = \begin{cases} 
\infty & \text{if } x > 0, \\
-\infty & \text{if } x \leq 0.
\end{cases}$$

We have

$$f^*(y) = \infty, \quad \forall \ y \in \mathbb{R}^n,$$

$$f^{**}(x) = -\infty, \quad \forall \ x \in \mathbb{R}^n.$$

But

$$\tilde{\text{cl}} \ f = f,$$

so $\tilde{\text{cl}} \ f \neq f^{**}$. 
A FEW EXAMPLES

- $l_p$ and $l_q$ norm conjugacy, where $\frac{1}{p} + \frac{1}{q} = 1$

  $$f(x) = \frac{1}{p} \sum_{i=1}^{n} |x_i|^p, \quad f^*(y) = \frac{1}{q} \sum_{i=1}^{n} |y_i|^q$$

- Conjugate of a strictly convex quadratic

  $$f(x) = \frac{1}{2} x'Qx + a'x + b,$$

  $$f^*(y) = \frac{1}{2} (y - a)'Q^{-1}(y - a) - b.$$  

- Conjugate of a function obtained by invertible linear transformation/translation of a function $p$

  $$f(x) = p(A(x - c)) + a'x + b,$$

  $$f^*(y) = q((A')^{-1}(y - a)) + c'y + d,$$

  where $q$ is the conjugate of $p$ and $d = -(c'a + b)$.  


SUPPORT FUNCTIONS

- Conjugate of indicator function $\delta_X$ of set $X$

$$\sigma_X(y) = \sup_{x \in X} y'x$$

is called the support function of $X$.

- To determine $\sigma_X(y)$ for a given vector $y$, we project the set $X$ on the line determined by $y$, we find $\hat{x}$, the extreme point of projection in the direction $y$, and we scale by setting

$$\sigma_X(y) = \|\hat{x}\| \cdot \|y\|$$

- $\text{epi}(\sigma_X)$ is a closed convex cone.

- The sets $X$, $\text{cl}(X)$, $\text{conv}(X)$, and $\text{cl}(\text{conv}(X))$ all have the same support function (by the conjugacy theorem).
SUPPORT FN OF A CONE - POLAR CONE

• If $C$ is a cone,

$$\sigma_C(y) = \sup_{x \in C} y'x = \begin{cases} 0 & \text{if } y'x \leq 0, \ \forall \ x \in C, \\ \infty & \text{otherwise} \end{cases}$$

i.e., $\sigma_C$ is the indicator function $\delta_{C^*}$ of the polar cone of $C$, given by

$$C^* = \{ y \mid y'x \leq 0, \ \forall \ x \in C \}$$

• By the Conjugacy Theorem the polar cone of $C^*$ is cl(conv($C'$)). This is the Polar Cone Theorem.

• Farkas’ Lemma: Let $a_1, \ldots, a_r$ be vectors in $\mathbb{R}^n$. Then the convex cone $C = \text{cone}\{a_1, \ldots, a_r\}$ is closed and its polar cone,

$$C^* = \{ x \mid a'_jx \leq 0, \ j = 1, \ldots, r \},$$

satisfies $(C^*)^* = C$.

Proof: For an elementary proof that $C$ is closed, see the Ch. 1 exercises. A more sophisticated proof is based on Prop. 1.4.13 of the text. The relation $(C^*)^* = C$ follows from the Polar Cone Theorem. Q.E.D.