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Psychoacoustics Project Laboratory

Notes on the Decision Model.

Shift Densities,
Gaussian Decision Variables: ROCs,
Psychometric Functions,
Evidence for the Decision Model Problems *

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5 Gaussian Densities

So far, we have made no assumptions in the model about the form of the densities $p_X (X_0 | S_j)$. We will now specify the model further by assuming that these densities are Gaussian. The assumption that the densities $p_X (X_0 | S_j)$ are Gaussian is reasonable if we interpret $X$ as the sum of a large number of similar, essentially independent, random variables and assume that some form of the Central Limit Theorem is applicable. Moreover, the Gaussian assumption, combined with the assumption of equal variances, makes the model extremely simple mathematically. See Appendix A for an introduction to Gaussian random variables.

5.1 Equal Variance Case

![Figure 7: A pair of Gaussian probability densities with $m_1 = 0$, $m_2 = 2$ and $\sigma_1 = \sigma_2 = 1$.](image)

For any pair of Gaussian densities that have the same variance, $\sigma^2$,

$$p_X (X_0 | S_1) = g_X \left( M_1, \sigma^2 \right)$$

$$p_X (X_0 | S_2) = g_X \left( M_2, \sigma^2 \right)$$

one always has, for any $K$,

$$p_X \left( \frac{M_1 + M_2}{2} + K | S_1 \right) = p_X \left( \frac{M_1 + M_2}{2} - K | S_2 \right)$$

(52)
Fig. 7 illustrates the case of two Gaussian densities which both have $\sigma = 1$. For these densities $m_1 = 0$ and $m_2 = 2$.

Consider now the parameters $d'$ and $\beta$ defined by the equations

\begin{align}
  d' &= \frac{M_2 - M_1}{\sigma} \\
  \beta &= \frac{1}{\sigma} \left[ C - \frac{M_2 + M_1}{2} \right]
\end{align}

The parameter $d'$, which is independent of the criterion $C$ and measures the separation of the densities $p_X(X_0|S_1)$ and $p_X(X_0|S_2)$ and, therefore, the distinguishability of $S_1$ and $S_2$, is referred to as the “sensitivity index” or merely “sensitivity.” It will be used extensively throughout these notes. The parameter $\beta$, which is a normalized form of the criterion $C$ and measures the tendency to respond $R_1$ rather than $R_2$, is referred to as the “bias.” The normalization has been chosen so that $\beta = 0$ when $C = \frac{M_1 + M_2}{2}$, that $\beta = \frac{1}{2}d'$ when $C = M_2$ and $\beta = -\frac{1}{2}d'$ when $C = M_1$.

Figure 8: The ROC curve for a pair of Gaussian probability densities with $m_1 = 0$, $m_2 = 2$ and $\sigma_1 = \sigma_2 = 1$.

Fig. 8 illustrates the ROC curve for the densities represented in Fig. 7.
For these Gaussian random variables

\[
P_F = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{(x-M_1)^2}{2\sigma^2}} \, dx = \frac{1}{\sqrt{2\pi}} \int_{\frac{C-M_1}{\sigma}}^{+\infty} e^{-\frac{x^2}{2}} \, dx
\]

\[
= 1 - \Phi\left(\frac{C-M_1}{\sigma}\right) = \Phi\left(\frac{M_1-C}{\sigma}\right)
\]

\[
P_D = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{(x-M_2)^2}{2\sigma^2}} \, dx = \frac{1}{\sqrt{2\pi}} \int_{\frac{C-M_2}{\sigma}}^{+\infty} e^{-\frac{x^2}{2}} \, dx
\]

\[
= 1 - \Phi\left(\frac{C-M_2}{\sigma}\right) = \Phi\left(\frac{M_2-C}{\sigma}\right)
\]

(55)

Consider now the transformation of the coordinates \((P_F, P_D)\) into the coordinates \((Z_F, Z_D)\) defined by the integral equations:

\[
P_F = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z_F} e^{-\frac{x^2}{2}} \, dx
\]

(57)

\[
P_D = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z_D} e^{-\frac{x^2}{2}} \, dx.
\]

(58)

The quantities \(Z_F\) and \(Z_D\) are the “normal deviates” of \(P_D\) and \(P_F\) and the pair \((Z_F, Z_D)\) is referred to as the “normal-deviate coordinate system” [as opposed to \((P_F, P_D)\), which is referred to as the “linear coordinate system”]. By combining equations (Eq. 6, 7, 39, 58, and 58 and making the appropriate substitution in the integrals), it is easy to show that if \(P_D(C)\) and \(P_F(C)\) are the probabilities associated with the criterion \(C\), then,

\[
Z_F = \frac{M_1-C}{\sigma}
\]

(59)

\[
Z_D = \frac{M_2-C}{\sigma}
\]

(60)

By combining Eqs. 54, 54, 60 and 60, we obtain the following relations between \((Z_F, Z_D)\) and \((d', \beta)\):

\[
d' = Z_D - Z_F
\]

(61)

\[
\beta = -\frac{Z_D + Z_F}{2}
\]

(62)

\[
Z_D = \frac{d'}{2} - \beta
\]

(63)

\[
Z_F = -\left(\frac{d'}{2} + \beta\right)
\]

(64)
The transformation between \( P_F, P_D \) and \( d', \beta \) is obtained by combining Eqs. 58, 58, 63, and 64.

The usefulness of the normal-deviate system follows from the fact that the contours of constant \( d' \) (i.e., the ROC’s or “iso-sensitivity curves”) and the contours of constant \( \beta \) (i.e., the “iso-bias curves”) are straight lines on these coordinates. Specifically, we see that

\[
\begin{align*}
Z_D &= Z_F + d' \\
Z_D &= -Z_F - 2\beta
\end{align*}
\]  

Thus, on normal-deviate coordinates, contours of constant \( d' \) are linear with slope 1 and intercept \( d' \), and the contours of constant \( \beta \) are linear with slope -1 and intercept -2\( \beta \). On linear coordinates, the contours are similar in that contours of constant \( d' \) are symmetric about the negative diagonal and the contours of constant \( \beta \) are symmetric about the positive diagonal, but all the contours (except for the \( d' = 0 \) and \( \beta = 0 \) contours) are curvilinear. A sheet of graph paper that makes use of the normal-deviate coordinates, and also provides a direct translation between \( P_D, P_F \) and \( Z_D, Z_F \), is attached to this manuscript. Fig. 9

![ROC curve](image)

**Figure 9:** The ROC curve, plotted on normal deviate coordinates, for a pair of Gaussian probability densities with \( m_1 = 0, \ m_2 = 2 \) and \( \sigma_1 = \sigma_2 = 1 \).

illustrates the ROC curve in z-space for the densities represented in Fig. 7.
6 ROC Experiment

In a classic experiment, Egan, Schulman, and Greenberg\textsuperscript{4} explored the ROC curves associated a one-interval experiment that measured with the detectability of a 500 ms tone that was turned on and off abruptly. Observation intervals occurred every 6.5 sec. The tone was added to a 500 ms burst of “white” noise of overall power of 65 dB SPL (spectrum level roughly $N_0 = 25$ dB SPL/Hz. No feedback was provided to the listeners as to the correctness of their responses.

In separate sessions, data were obtained by the binary-decisions (signal–no signal) procedure or by the rating method. In the binary-decision procedure, listeners were trained to adopt three different criteria: “strict”, “medium”, and “lax”. In the rating method, listeners assigned one of four responses to each stimulus: with a “1” corresponding to a “signal-present” response with a strict criterion, a “2” to a “signal-present” response with a medium criterion, a “3” to a “signal-present” response with a lax criterion, and a “4” to a “signal-absent” response with a lax criterion.

Three listeners were tested in nine daily sessions: days 1–3 and 7–9 were devoted to binary-decisions, day 4 to practice on the rating method and days 5–6 to tests with that method. Each daily session consisted of 9 test periods of 80 trials each (8.7 minutes) separated by short rest intervals. For the binary decision method, 240 trials were devoted to each of the three criteria in random order. A total of 4320 trials were conducted using the binary-decision method and 1440 with the rating method.

In this experiment the probability of presenting a signal was one-half and the ratio the signal-energy $E$ (time-integral of signal power) to the the power spectral density of the noise $N_0$ was:

$$\frac{E}{N_0} = 15.8$$

$$10 \log_{10} \frac{E}{N_0} = 12 \text{ dB}$$

Data were analyzed by plotting points on an ROC, each corresponding to 240 trials. Least-squares best-fit straight lines were fit to normal-normal ROC plots with the “x-intercept” taken as an estimate of $d'$ and the reciprocal of the slope taken to as an estimate of $\sigma_{SN}/\sigma_N$ (Fig. 14). For the three listeners tested, $d' = 1.30, 1.52,$ and 1.85 for the binary decision method and $d' = 1.42, 1.36,$ and 1.82 for the rating method. the corresponding values of $\sigma_{SN}/\sigma_N$ were 1.03, 1.06, and 1.36 for the binary-decision method, and 1.31, 1.13, and 1.06 for the rating method. The authors concluded that trained listeners were able to perform as well when they adopted the multiple criteria needed for the rating method as when they adopted the single criterion required for the binary-decision procedure.

Figure 8: The results of the experiment with a fixed $E/N_0 = 15.8$. The three panels on the left are for the binary-decision method. Those on the right are for the rating method.
7 Bias

![Bias](image.png)

Figure 9: The results of the experiment with five values of a priori probability: $P(S_2) = 0.1, 0.3, 0.5, 0.7,$ and 0.9 displayed as an ROC curve. The experiment was presented twice.

We showed in Sec. ?? that the observer should

$$\frac{p_X(X_0|S_1)}{p_X(X_0|S_2)} \geq \frac{(V_{22} - V_{21})P_2}{(V_{11} - V_{12})P_1}$$

In the case where:

$$p_X(X_0|S_2) = g_X(M_2, \sigma)$$

$$p_X(X_0|S_1) = g_X(M_1, \sigma),$$

it can be shown that this decision rule is equivalent to

$$\frac{1}{\sigma} \left( X_0 - \frac{M_2 + M_1}{2} \right) \geq \frac{1}{d'} \ln \frac{P_2 V_{22} - V_{21}}{P_1 V_{11} - V_{12}}.$$

which, in the case where the payoffs term is equal to one, becomes

$$\frac{1}{\sigma} \left( X_0 - \frac{M_2 + M_1}{2} \right) \geq \frac{1}{d'} \ln \frac{P_2}{P_1}.$$

In a classic experiments of Tanner, Swets, and Green (1956)\(^5\) had one subject detect a 100 ms burst of 1000 Hz. tone in a continuous “white” noise background, with lights marking

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the observation intervals. There were two responses “signal” ($R_2$) and “no-signal” ($R_1$). Five levels of a priori probability of presenting the signal $P_2 = 0.1, 0.3, 0.5, 0.7,$ and 0.9 were employed, one in each session of 300 trials. After each trial, the listener was informed of the correctness of his response. The subject was awarded a fraction of a cent for each correct response and fined an equal amount for each incorrect response.

Tanner et al. analyzed the results in terms of an equal-variance Gaussian model. Their finding indicated that $d' = 0.92$ (Fig. ??).

![Figure 10: The results of the experiment with five values of a priori probability: $P(S_2) = 0.1, 0.3, 0.5, 0.7,$ and 0.9 and the actual odds of saying $R_2 = N_2/N$.](image)

Tanner et al. computed the Observed Odds of saying $R_2, N_2/N$, and compared that against the Actual Odds, $P_2/P_1$ (Fig. 10). They found that the Observed Odds always to be closer to unity than the Actual Odds. When the a priori probability is 0.1, listeners behave as if it were roughly 0.3. When the a priori probability is 0.9, listeners behave as if it were roughly 0.7. This reflects a general tendency to “underestimate the odds”. We are unlikely to respond appropriately when the odds in favor of an unlikely event (such as an airplane crash) are $1,000,000:1$. We tend to equate them with the odds of an accident driving to the airport ($10,000:1$).
Figure 11: The results of the experiment with five values of *a priori* probability: $P(S_2) = 0.1, 0.3, 0.5, 0.7,$ and $0.9$ and the actual odds of saying $R_2 = N_2/N$. 
Figure 12: The results of the experiment with five values of a priori probability: $P(S_2) = 0.1, 0.3, 0.5, 0.7$, and 0.9 and the actual odds of saying $R_2 = N_2/N$. 
8 Gaussian Psychometric Functions

![Graph showing the relationship between $d'$ and $E/N_0$.](image)

Figure 13: The results of the experiment with three values $E/N_0$. Average estimates of $d'$ from each of three listeners are plotted as a function of $d'$ and fit to a straight line.

In a second experiment, the same three listeners tested in Sec. 6 were tested using the rating method to determine the relation between $d'$ and $E/N_0$. Listeners JE and AS were tested at $E/N_0 = 7.9, 15.8,$ and $31.6$. PE was tested at $E/N_0 = 6.2, 12.6$ and $25.1$. Each subject participated in 6 sessions, two at each value $E/N_0$. Sessions consisted of 240 practice trials and 480 test trials, from which data were collected.

ROC curves on normal-normal coordinates are shown in Fig. ?? together with estimates of $d'$ and $\sigma_{SN}/\sigma_N$. The values of $d'$ are plotted as a function of $E/N_0$ in Fig. ??, and fit to a straight line

$$d' = 0.095 \frac{E}{N_0}.$$
Figure 14: The results of the experiment with three values $E/N_0$. The three panels are the three listeners who participated in tests that used the rating method.
A Gaussian Random Variables

In this section we summarize some properties of Gaussian (or Normal) random variables.

\[ p_X(X_0) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(X_0-M)^2}{2\sigma^2}} \]  (39)

It is easy to prove that

\[ E[X] = M \]  (40)
\[ E[(X-M)^2] = \sigma^2 \]  (41)

thus the mean of \( X \) is \( M \) and the variance of \( X \) is \( \sigma^2 \). A picture of a Gaussian density (with \( M = 0 \) and \( \sigma = 1 \)) is shown in Fig. 15. As can be seen, the density is symmetric about its mean

\[ p_X(M + K) = p_X(M - K) \]
Increasing $\sigma$ widens and flattens the density function.

The unit Gaussian density has a mean of 0 and a variance of 1.

$$G_X (X_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

It is easy to show that for any Gaussian density $p_X$ with mean $M$ and variance $\sigma^2$

$$p_X (X_0) = g_X (M, \sigma^2) = \frac{1}{\sigma} G_X \left( \frac{X_0 - M}{\sigma} \right)$$

![Graph of the Gaussian probability density function]

Figure 16: Integral of the Gaussian probability density.

The cumulative distribution function of the unit Gaussian random variable is $\Phi (x_0) = \Pr (x \leq x_0)$

$$\Phi (x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

Note that $\Phi (x) = 1 - \Phi (-x)$. Unfortunately there is no closed form analytical expression for the integral of the Gaussian density between finite limits. Very good approximations to the integral of the unit Gaussian density are available, e.g.

$$\Phi (x) \approx 1 - \left( a_1 t + a_2 t^2 + a_3 t^3 \right) \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$
where
\[ t = \frac{1}{1 + px} \]

and
\[ p = +0.332670 \]
\[ a1 = +0.4361836 \]
\[ a2 = -0.1201676 \]
\[ a3 = +0.9372980 \]

The maximum absolute error in using the approximation is roughly 0.00001 for 0 ≤ x ≤ 5 and the maximum relative error (in comparison to 1 − Φ (x)) is roughly 0.0075 for 0 ≤ x ≤ 5.

Table ?? lists the value of the Φ function. It should be noted that
\[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-M)^2}{2\sigma^2}} \, du = \Phi \left( \frac{x_0 - M}{\sigma} \right) \]

Note that it is possible to express
\[ \Pr (a \leq x < b) = \Phi (b) - \Phi (a) \] (42)

The commonly available “error function” erf (x) defined as
\[ \text{erf} (x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \, dt \] (43)

is related to the cumulative distribution function Φ by
\[ \Phi (x) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{x}{\sqrt{2}} \right) \] (44)

The inverse of the Φ function is the z-score function, z (p) defined implicitly by
\[ p = \int_{-\infty}^{z(p)} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du \]

Note that \( z (p) = -z (1 - p) \).

Again, there is no closed form analytical expression for the z-score function. Very good approximations to this function are available, e.g., when \( 0 < p \leq 0.5 \)
\[ z (p) \approx -t + \frac{c_0 + c_1 t + c_2 t^2}{d_0 + d_1 t + d_2 t^2 + d_3 t^3} \]
where
\[ t = \sqrt{-2 \ln p} \]

and
\( c_0 = 2.515517 \quad d_0 = 1.000000 \)
\( c_1 = 0.802853 \quad d_1 = 1.432788 \)
\( c_2 = 0.010328 \quad d_2 = 0.189269 \)
\( d_3 = 0.001308 \)

The \( z \)-score function is illustrated in Fig. 17.

![Z-score function](image)

**Figure 17:** The \( z \)-score function.

### A.2 Law of Large Numbers

When the number of independent, identically distributed, finite variance, random variables, \( n \) is sufficiently large, the central limit theorem applies to their sum, \( S \).

\[
\Pr(S \leq S_0) \approx \frac{1}{2\pi \sigma_S} \int_{-\infty}^{S_0} \exp\left\{ -\frac{(x - \mu_S)^2}{2\sigma_S^2} \right\} dx
\]

where \( \mu_S \) is the expected value of \( S \), and \( \sigma_S^2 \) is the variance of \( S \).

This formula can also be used to approximate the probability that \( S \) falls within limits,
\[
[S_1, S_2]
\]

\[
\Pr(S_1 \leq S \leq S_2) \approx \frac{1}{2\pi \sigma_S} \int_{S_1}^{S_2} \exp \left\{ -\frac{(x - \mu_S)^2}{2\sigma_S^2} \right\} dx
\]

If the random variables are coin flips, the cumulative distribution corresponding to \(\Pr(h|n, p_H)\) approaches the cumulative distribution of a Gaussian random variable, with the same mean and variance. We indicate this, loosely, by

\[
\Pr(h|n, p_H) \rightarrow g(np_H, np_H (1 - p_H))
\]

where \(g(\mu, \sigma^2)\) denotes a Gaussian probability density with mean \(\mu\) and variance \(\sigma^2\).