Lecture Outline

- Nonlinear Programming
- Application Contexts
- Characterization Issue
- Computation Issue
- Duality
- Organization
Mathematical Formulation of an Optimization Problem

minimize \( f(x) \)
subject to \( x \in X \)

- Set \( X \subseteq \mathbb{R}^n \) represents the constraint (feasible) set
- Vector \( x = (x_1, \ldots, x_n) \) represents optimization (decision) variables
- Function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is the objective (cost) function
  - \( f \) is assumed to be a continuous (and usually differentiable) function
- Optimal value: Smallest value of \( f \) among all feasible vectors
- Optimal solution: A feasible vector that achieves the optimal value
Classes of Optimization Problems

- **Discrete Optimization Problems:** $X$ is finite
  - Examples: combinatorial problems (scheduling, matching), integer programming (variables constrained to be integer)

- **Continuous Optimization Problems:**
  - If $X = \mathbb{R}^n$, the problem is an unconstrained optimization prob.
  - If $f$ is linear and $X$ is polyhedral, the problem is a linear programming prob. O/w it is a nonlinear programming prob.
  - Linear/nonlinear programming have traditionally been treated separately. Their methodologies have gradually come closer.
  - If $f$ and $X$ are convex, the problem is a convex optimization problem (much recent interest and many applications)

- **Our focus:** Nonlinear optimization problems
  - $X = \mathbb{R}^n$ or $X \subset \mathbb{R}^n$ (specified by equations and inequalities) with a continuous character
  - Also cover convex optimization problems and methodology
Applications of Nonlinear Programming

- Communication and transportation networks (e.g., routing, flow control)
- Data analysis and least squares formulations
- Estimation, signal and image processing
- Pricing and resource allocation
- Solution of equilibrium models
- Finance
- Machine learning
Two main issues

• Existence and characterization of minima
  – Necessary conditions
  – Sufficient conditions
  – Lagrange multiplier theory
  – Sensitivity
  – Duality

• Computation by iterative algorithms
  – Iterative descent
  – Penalty and barrier methods
  – Interior point methods
  – Subgradient methods (for convex and nonsmooth problems)
Characterization problem

• Unconstrained problems
  – Zero 1st order variation along all directions

• Constrained problems
  – Nonnegative 1st order variation along all feasible directions

• Equality constraints
  – Zero 1st order variation along all directions on the constraint surface
  – Lagrange multiplier theory

• Sensitivity
Computation problem

• Iterative descent
• Approximation
• Role of convergence analysis
• Role of rate of convergence analysis
• Using an existing package to solve a nonlinear programming problem
Illustration of the optimal values of the min common point and max intercept point problems. In (a), optimal values are not equal. In (b), the set $S$, when “extended upwards” along the $n$th axis, yields

$$\tilde{S} = \{\tilde{x} \mid \text{for some } x \in S, \; \tilde{x}_n \geq x_n, \; \tilde{x}_i = x_i, \; i = 1, \ldots, n - 1\},$$

which is convex. As a result, the two optimal values are equal. This fact, when suitably formalized, is the basis for some of the most important duality results.
Convex Optimization

• Convexity theory and analysis have been studied for a long time

• Until late 1980’s:
  – Algorithmic development focused mainly on solving linear probs
    ∗ Simplex method for linear programming (Dantzig, 1947)
    ∗ Ellipsoid method (Shor, 1970; Khachiyan, 1979)
    ∗ Interior point methods for LP (Karmarkar, 1984)

• Since late 1980’s: New interest in convex optimization
  – The recognition that Interior point methods can efficiently solve convex problems, including semidefinite programs and second-order cone programs, *almost as easily as linear programs*
  – Convex problems prevalent in practice, also convex problems provide systematic relaxations for hard problems.

• More recently, motivated by very large scale structured problems, a resurgence of first-order (gradient-based) methods.
Organization of the Course

- Unconstrained optimization (Theory and Algorithms) - 5 Lectures
- Optimization over a convex set - 2 Lectures
- Lagrange Multiplier Theory - 3 Lectures
- Convex optimization and duality - 3 Lectures
- Conic and semidefinite optimization - 2 Lectures
- Interior point methods - 3 Lectures
- Dual methods (subgradient methods for convex nonsmooth problems) - 3 Lectures
- Nonsmooth optimization, variational inequalities - 2 Lectures
- Review and applications - 2 Lectures