Lecture Outline

• The Dual Problem and Properties

• Strong Duality Theorems

Consider the problem

\[ f^* = \text{minimize} \quad f(x) \]

subject to \( x \in X, \quad g_j(x) \leq 0, \quad j = 1, \ldots, r \)

We assume that the problem is feasible and the cost is bounded from below: \( -\infty < f^* < \infty \)

• \( \mu^* \) is a multiplier if \( \mu^* \geq 0 \) and \( f^* = \inf_{x \in X} L(x, \mu^*) \).
The dual problem is:

\[
\text{maximize}_{\mu \geq 0} \, q(\mu),
\]

where \( q \) is the dual function

\[
q(\mu) = \inf_{x \in X} L(x, \mu), \quad \forall \mu \in \mathbb{R}^r.
\]

Question: How does the optimal dual value \( q^* = \sup_{\mu \geq 0} q(\mu) \) relate to \( f^* \)?

Support points correspond to minimizers of \( L(x, m) \) over \( X \)

\[H = \{(z,w) \mid w + m'z = b\}\]

\[S = \{(g(x), f(x)) \mid x \in \text{EX}\}\]

Optimal Dual Value
Weak Duality

• The domain of $q$ is

$$D_q = \{\mu \mid q(\mu) > -\infty\}.$$

• **Proposition:** The domain $D_q$ is a convex set and $q$ is concave over $D_q$.

• **Proposition:** (Weak Duality Theorem) We have

$$q^* \leq f^*.$$

**Proof:** For all $\mu \geq 0$, and $x \in X$ with $g(x) \leq 0$, we have

$$q(\mu) = \inf_{z \in X} L(z, \mu) \leq f(x) + \sum_{j=1}^{r} \mu_j g_j(x) \leq f(x),$$

so

$$q^* = \sup_{\mu \geq 0} q(\mu) \leq \inf_{x \in X, g(x) \leq 0} f(x) = f^*.$$
Dual Optimal Solutions and Multipliers

Proposition: (a) If $q^* = f^*$, the set of multipliers is equal to the set of optimal dual solutions. (b) If $q^* < f^*$, the set of multipliers is empty.

Proof: By definition, a vector $\mu^* \geq 0$ is a multiplier if and only if $f^* = q(\mu^*) \leq q^*$, which by the weak duality theorem, holds if and only if there is no duality gap and $\mu^*$ is a dual optimal solution.
Duality Properties

- **Optimality Conditions:** \((x^*, \mu^*)\) is an optimal solution-multiplier pair if and only if

\[
x^* \in X, \quad g(x^*) \leq 0, \quad \mu^* \geq 0,
\]

(Primal Feasibility),

\[
x^* = \arg \min_{x \in X} L(x, \mu^*),
\]

(Dual Feasibility),

\[
\mu_j^* g_j(x^*) = 0, \quad j = 1, \ldots, r,
\]

(Lagrangian Optimality),

(Complementary Slackness).

- **Saddle Point Theorem:** \((x^*, \mu^*)\) is an optimal solution-multiplier pair if and only if \(x^* \in X, \mu^* \geq 0\), and \((x^*, \mu^*)\) is a saddle point of the Lagrangian, in the sense that

\[
L(x^*, \mu) \leq L(x^*, \mu^*) \leq L(x, \mu^*), \quad \forall x \in X, \mu \geq 0.
\]
Strong Duality Theorem I

- Theorem (Convex Objective-Linear Constraints):

  \[
  \begin{aligned}
  &\text{minimize} & & f(x) \\
  &\text{subject to} & & x \in X, \quad a_i'x - b_i = 0, \quad i = 1, \ldots, m, \\
  & & & e_j'x - d_j \leq 0, \quad j = 1, \ldots, r,
  \end{aligned}
  \]

  Assume that \( f^* \) is finite. Let also \( f \) be convex over \( \mathbb{R}^n \) and let \( X \) be polyhedral. Then:
  
  - There is no duality gap, i.e., \( q^* = f^* \).
  - There exists a dual optimal solution.

- Proof Approach:

  - Existence of primal optimal solutions of LPs
  - Extended Farkas Lemma

- Convexity of \( f \) over the entire space \( \mathbb{R}^n \) is essential.
Convex-Linear Problem with a Duality Gap

• Consider the two-dimensional problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x_1 \leq 0, \quad x \in X = \{x \mid x \geq 0\},
\end{align*}
\]

where

\[
f(x) = e^{-\sqrt{x_1 x_2}}, \quad \forall \ x \in X,
\]

and \( f(x) \) is arbitrarily defined for \( x \notin X \).

• \( f \) is convex over \( X \) (composition of a monotonically increasing convex function and a convex function), and \( f^* = 1 \).

• Also, for all \( \mu \geq 0 \) we have

\[
q(\mu) = \inf_{x \geq 0} \{e^{-\sqrt{x_1 x_2}} + \mu x_1\} = 0,
\]

since the expression in braces is nonnegative for \( x \geq 0 \) and can approach zero by taking \( x_1 \rightarrow 0 \) and \( x_1 x_2 \rightarrow \infty \). It follows that \( q^* = 0 \).
Strong Duality Theorem II

minimize $f(x)$
subject to $x \in X$, $g_j(x) \leq 0$, $j = 1, \ldots, r$

• Assume that $X$ is convex and the functions $f : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}$ are convex over $X$. Also assume that $f^*$ is finite.

• **Slater condition:** There exists a vector $\bar{x} \in X$ such that

$$g_j(\bar{x}) < 0, \quad \forall \ j = 1, \ldots, r.$$

The Slater condition is a condition on the constraint set only.

• **Theorem (Convex Objective-Convex Inequality Constraints):** Assume that Slater condition holds. Then:
  - There is no duality gap, i.e., $q^* = f^*$.
  - The set of dual optimal solutions is nonempty and bounded.
Visualization of the Slater Condition

Consider the set $A \subset \mathbb{R}^r \times \mathbb{R}$ given by

$$A = \{(z, w) \mid \text{there is some } x \in X \text{ such that } g(x) \leq z, \ f(x) \leq w\}$$
Proof

• By the convexity of $f$, $g_j$’s, and $X$, the set $A$ is convex.

• The vector $(0, f^*)$ is not in the interior of the set $A$. Assume it is, i.e., $(0, f^*) \in \text{int}(A)$. Then, there exists an $\epsilon > 0$ such that $(0, f^* - \epsilon) \in A$, contradicting the optimality of $f^*$.

• Thus, either $(0, f^*) \in \text{bd}A$ or $(0, f^*) \notin A$.

• By the Supporting Hyperplane Theorem, there exists a hyperplane passing through $(0, f^*)$ and containing $A$ in one of the two corresponding halfspaces; i.e., there exists $(\mu, \beta) \neq (0, 0)$ with

$$\beta f^* \leq \beta w + \mu^t z, \quad \forall (z, w) \in A.$$

This implies that $\beta \geq 0$, and $\mu_j \geq 0$ for all $j$.

• Prove that the hyperplane is nonvertical, i.e., $\beta > 0$. 
• Normalize ($\beta = 1$), take the infimum over $x \in X$, and use the fact $\mu \geq 0$, to obtain

$$f^* \leq \inf_{x \in X} \{ f(x) + \mu' g(x) \} = q(\mu) \leq \sup_{\bar{\mu}\geq0} q(\bar{\mu}) = q^*.$$  

Using the weak duality theorem, there is no duality gap and $\mu$ is a dual optimal solution.

• We finally show that the set of dual optimal solutions is bounded. For any dual optimal solution $\mu \geq 0$, we have

$$q^* = q(\mu) = \inf_{x \in X} \{ f(x) + \mu' g(x) \}$$

$$\leq f(\bar{x}) + \mu' g(\bar{x})$$

$$\leq f(\bar{x}) + \max_{1 \leq j \leq r} \{ g_j(\bar{x}) \} \sum_{j=1}^{r} \mu_j.$$
Therefore, \( \min_{1 \leq j \leq r} \{-g_j(\bar{x})\} \sum_{j=1}^{r} \mu_j \leq f(\bar{x}) - q^* \) showing that

\[
\sum_{j=1}^{r} \mu_j \leq \frac{f(\bar{x}) - q^*}{\min_{1 \leq j \leq r} \{-g_j(\bar{x})\}}
\]

Q.E.D.

- The converse statement is also true, i.e., if the set of Lagrange multipliers is nonempty and bounded, then the Slater condition holds.
  - Proof is left for the homework
Linear Equality Constraints

• Suppose we have the additional constraints

\[ e_i'x - d_i = 0, \quad i = 1, \ldots, m \]

• We need the notion of the **affine hull** of a convex set \( X \) [denoted \( \text{aff}(X) \)]. This is the intersection of all affine sets containing \( X \).

• The **relative interior** of \( X \), denoted \( \text{ri}(X) \), is the set of all \( x \in X \) s.t. there exists \( \epsilon > 0 \) with

\[ \{ z \mid \|z - x\| < \epsilon, \ z \in \text{aff}(X) \} \subset X, \]

that is, \( \text{ri}(X) \) is the interior of \( X \) relative to \( \text{aff}(X) \).

• **Theorem:** Every nonempty convex set has a nonempty relative interior.
Strong Duality Theorem for Equalities

• **Assumptions:**
  
  – The set $X$ is convex and the functions $f$, $g_j$ are convex over $X$.
  
  – The optimal value $f^*$ is finite and there exists a vector $\bar{x} \in \text{ri}(X)$ such that

    $$g_j(\bar{x}) < 0, \quad j = 1, \ldots, r,$$

    $$e'_i \bar{x} - d_i = 0, \quad i = 1, \ldots, m.$$ 

• Under the preceding assumptions there is no duality gap and a dual optimal solution exists.
Counterexample

• Consider

\[
\begin{align*}
\text{minimize} & \quad f(x) = x_1 \\
\text{subject to} & \quad x_2 = 0, \quad x \in X = \{(x_1, x_2) \mid x_1^2 \leq x_2\}.
\end{align*}
\]

• The optimal solution is \( x^* = (0, 0) \) and \( f^* = 0 \).

• The dual function is given by

\[
q(\lambda) = \inf_{x_1^2 \leq x_2} \{x_1 + \lambda x_2\} = \begin{cases} 
-\frac{1}{4\lambda} & \text{if } \lambda > 0 \\
-\infty & \text{if } \lambda \leq 0.
\end{cases}
\]

• No dual optimal solution and therefore there is no multiplier. (Even though there is no duality gap.)

• Assumptions are violated (the feasible set and the relative interior of \( X \) have no common point).
Quadratic Objective-Quadratic Constraints

\begin{align*}
\text{minimize} & \quad x'Q_0x + a'_0x + b_0 \\
\text{subject to} & \quad x'Q_jx + a'_jx + b_j \leq 0, \quad j = 1, \ldots, r
\end{align*}

**Theorem:** Assume that each $Q_i$ is a symmetric positive semidefinite matrix. Assume that $f^*$ is finite. Then, there is no duality gap and there exists a **primal** solution.

**Remarks:**

- The theorem says nothing about the existence of dual optimal solutions.
- The theorem extends to convex polynomial objective and convex polynomial constraints.
- **Open problem:** The case of convex smooth objective and convex smooth constraints!!