Lecture Outline

• Conic optimization: background and formulation
• Modelling aspects
• Main examples: linear, second-order, semidefinite
• Conic duality
• Applications
Motivation and advantages

- Elegant and symmetric formulation
- Generalization of linear programming
- Some constraints can be naturally expressed in conic form, and are awkward or inconvenient as explicit inequalities
- Enables the development of specialized algorithms (e.g., interior point)
- Focus on geometry, not on the representation
Review: Proper Cones

A cone is a set $K$ that is closed under nonnegative scalings, i.e.,

$$x \in K \implies \lambda x \in K \quad \forall \lambda \geq 0$$

Important class: proper cones:

(a) Closed
(b) Convex
(c) Solid: full dimensional, there exists $B_\epsilon(x) \subset K$ for some $\epsilon > 0$
(d) Pointed: contains no lines, i.e., $K \cap (-K) = \{0\}$
General formulation

Let $K$ be a proper cone in $\mathbb{R}^n$, equipped with the inner product $\langle \cdot, \cdot \rangle$.

Define the conic programming problem:

$$\min_x \langle c, x \rangle \quad \text{s.t.} \quad Ax = b, \quad x \in K$$

- Geometrically, optimize a linear function over the intersection of an affine subspace and the cone $K$
- Description of $K$ not specified
- If $K$ is the nonnegative orthant, this is an LP problem
- In general, $K$ may not be polyhedral
Nonnegative orthant

• The nonnegative orthant $\mathbb{R}^n_+$ is

$$\{x \in \mathbb{R}^n \mid x_i \geq 0\}$$

• The feasible set is a polyhedron
• Optimization becomes a linear programming problem
Second-order cone

- The *second-order* cone $Q^n$ is defined as

$$\left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \leq t \right\}$$

- Also known as quadratic, Lorenz, or ice-cream cone
- For constant $t$, the sections are $n$-dimensional balls
- Not polyhedral
Semidefinite cone

- Space of symmetric matrices $S_n \approx \mathbb{R}^{\binom{n+1}{2}}$

- Trace $\text{Tr} X = \sum_i X_{ii} = \sum_i \lambda_i(X)$.

- Inner product: $\langle X, Y \rangle = \text{Tr} XY = \sum_{ij} X_{ij}Y_{ij}$

- (Frobenius) Norm: $\|X\| = \langle X, X \rangle^{\frac{1}{2}} = \left( \sum_{ij} X_{ij}^2 \right)^{\frac{1}{2}}$

- The **semidefinite** cone $S^n_+$ is defined as

$$\left\{ X \in S^n \mid v^T X v \geq 0 \quad \forall v \in \mathbb{R}^n \right\}$$

- Not polyhedral

- Contains $\mathbb{R}^n_+$ and $Q^n$ as affine sections
The cones $\mathbb{R}_+^n, \mathbb{Q}_+^n, \mathbb{S}_+^n$ are all proper.

Recall the general conic programming formulation:

$$\min_{x} \langle c, x \rangle \quad \text{s.t.} \quad Ax = b, \quad x \in K$$

For these specific choices of $K$, we obtain:

- $\mathbb{R}_+^n$: Linear programming (LP)
- $\mathbb{Q}_+^n$: Second-order cone programming (SOCP)
- $\mathbb{S}_+^n$: Semidefinite programming (SDP)

Also, direct products of cones (e.g., $\mathbb{R}_+^n \times \mathbb{S}_+^m$)

Many applications!
Example: facility location

Known as the *Fermat-Weber* problem.

- Given points \(y_1, \ldots, y_m\) in \(\mathbb{R}^n\), find a new point \(x\) that minimizes the sum of the distances to all the other points.
- Minimize \(\sum_{i=1}^{m} d_i\), subject to \(\|x - y_i\| \leq d_i\)
- Equivalently written as a conic SOCP problem:

\[
\min \sum_{i=1}^{m} d_i \quad \text{s.t.} \quad (x - y_i, d_i) \in Q^n
\]

- More “standard” SOCP form:

\[
\min \sum_{i=1}^{m} t_i \quad \text{s.t.} \quad x - x_i = y_i, \quad (x_i, t_i) \in Q^n
\]

The cone \(K\) is here \(\mathbb{R}^n \times Q^n \times \cdots \times Q^n\)
Dual cones

Given $K \subset \mathbb{R}^n$, the dual cone is

$$K^* := \{ y \in \mathbb{R}^n \mid y^T x \geq 0 \quad \forall x \in K \}$$

- Important case: self-dual cones ($K = K^*$)
- The cones $\mathbb{R}^n$, $Q^n$, $S^n_+$ are self-dual
- Also, many interesting non self-dual examples: $\ell_p$ norms, diagonally dominant matrices, nonnegative polynomials, etc.
Nonnegative polynomials

Univariate trigonometric polynomials:

\[ \{(p_0, \ldots, p_n) \in \mathbb{R}^{n+1} : p_0 + p_1 \cos \theta + \cdots + p_n \cos n\theta \geq 0 \quad \forall \theta \in [-\pi, \pi] \} \]

- This is a proper cone (why?)

- Example application: FIR filter design with interpolation constraints
Conic Duality

• Let $K$ be a proper cone, and $K^*$ its dual (which is also proper).

• Primal-dual pair of conic programs:

$$\begin{align*}
\min \quad & \langle c, x \rangle \\
\text{s.t.} \quad & Ax = b \\
& x \in K
\end{align*}$$

$$\begin{align*}
\max \quad & \langle b, y \rangle \\
\text{s.t.} \quad & c - A^T y \in K^*
\end{align*}$$

• Weak duality always holds: for any feasible $x, y$

$$\langle c, x \rangle - \langle b, y \rangle = \langle c, x \rangle - \langle Ax, y \rangle = \langle x, c - A^T y \rangle \geq 0$$
**Strong Duality**

Slater condition: existence of strictly feasible solution (interior point)

- If Slater holds for the primal (and primal is bounded below):
  - Dual optimal solutions exist
  - No duality gap (i.e., $p^* = d^*$)
  - Primal may not be attained

- If Slater holds for the primal and dual (i.e., there exists strictly feasible $x$ and $y$), then:
  - No duality gap ($p^* = d^*$)
  - Optimal primal and dual solutions exist
Strong duality can fail

- Duality is about strict separation of the set $AK$ and the point $b$

- If $AK$ is not closed, then strong duality can fail
  - Optimal value not achieved
  - Nonzero duality gaps (finite or infinite)

- Example in SOCP:

  \[
  \begin{align*}
  \min & -x_2 \\
  \text{s.t.} & -x_1 + t = 0, \quad (x_1, x_2, t) \in Q^2 \\
  \end{align*}
  \]
  
  - Feasible set is \( \{(x_1, x_2, x_1) \mid x_1 \geq 0, x_2 = 0\} \).
  
  - Primal Slater fails.

  - Primal optimal value is zero.

  - Dual is infeasible:

  \[
  \begin{align*}
  \max & 0 \\
  \text{s.t.} & (-y, -1, y) \in Q^2 \\
  \end{align*}
  \]
Interior point methods

- Algorithmic theory for conic optimization
- Relies on barrier functions with special properties (self-concordance)
- Builds on techniques we have seen (mostly, Newton’s method)
- For “nice” cones (e.g., LP, SOCP, SDP) yields polynomial-time convergence