1.4.5

Consider a truncated Newton method with the stepsize chosen by the Armijo rule with initial stepsize \( s = 1 \) and \( \sigma < 1/2 \), and assume that \( \{x^k\} \) converges to a nonsingular local minimum \( x^* \). Assume that the matrices \( H^k \) and the directions \( d^k \) satisfy

\[
\lim_{k \to \infty} \left\| H^k - \nabla^2 f(x^k) \right\| = 0, \quad \lim_{k \to \infty} \frac{\left\| H^k d^k + \nabla f(x^k) \right\|}{\left\| \nabla f(x^k) \right\|} = 0.
\]

Show that \( \{\|x^k - x^*\|\} \) converges superlinearly.

1.4.6

Apply Newton's method with a constant stepsize to minimization of the function \( f(x) = \|x\|^3 \). Identify the range of stepsizes for which convergence is obtained, and show that it includes the unit stepsize. Show that for any stepsize within this range, the method converges linearly to \( x^* = 0 \). Explain this fact in light of Prop. 1.4.1.

1.4.7

Consider Newton's method with the trust region implementation for the case of a positive definite quadratic cost function. Show that the method terminates in a finite number of iterations.

1.4.8

(a) Consider the pure form of Newton's method for the case of the cost function \( f(x) = \|x\|^\beta \), where \( \beta > 1 \). For what starting points and values of \( \beta \) does the method converge to the optimal solution? What happens when \( \beta \leq 1 \)?

(b) Repeat part (a) for the case where Newton's method with the Armijo rule is used.

1.4.9 (Computational Problem)

Consider a firm wishing to maximize its earnings by optimally choosing the selling price of its product, denoted \( y \), and the amount spent for advertising, denoted \( z \). Assume the following relationships:

\[
E = yx - (z + g_2(x)),
\]

\[
x = g_1(y, z) = a_1 + a_2y + a_3z + a_4yz + a_5z^2,
\]
\[ g_2(x) = e_1 + e_2 x, \]

where

\( E = \text{earnings} \)

\( x = \text{number of units sold} \)

\( y = \text{unit selling price} \)

\( z = \text{money spent for advertising} \)

\( g_1(y, z) = \text{predicted number of units sold when the price is } y \text{ and the cost of advertising is } z \)

\( g_2(x) = \text{production cost of } x \text{ units} \)

and the parameters \( a_i \) and \( e_i \) are

\[ a_1 = 50,000, \quad a_2 = -5,000, \quad a_3 = 40, \quad a_4 = -1, \quad a_5 = -0.002, \]

\[ e_1 = 100,000, \quad e_2 = 2. \]

Thus earnings can be expressed as a third degree polynomial of the two variables \( y \) and \( z \).

Find the values of \( y \) and \( z \) that minimize \(-E\) (or equivalently maximize the earnings \( E \) of the firm) by using steepest descent and Newton's method. Set up the methods in any way you prefer. Solve the problem as if it were unconstrained. Obtain the solution from several starting points, and experiment with stepsize selection and scaling techniques.

### 1.5 LEAST SQUARES PROBLEMS

In this section we consider methods for solving least squares problems of the form

\[
\text{minimize } f(x) = \frac{1}{2} \|g(x)\|^2 = \frac{1}{2} \sum_{i=1}^{m} \|g_i(x)\|^2 \tag{1.81}
\]

subject to \( x \in \mathbb{R}^n \),

where \( g \) is a continuously differentiable function with component functions \( g_1, \ldots, g_m \), where \( g_i : \mathbb{R}^n \to \mathbb{R}^{r_i} \). Usually \( r_i = 1 \), but it is sometimes notationally convenient to consider the more general case.

Least squares problems are very common in practice. An important case arises when \( g \) consists of \( n \) scalar-valued functions and we want to solve the system of \( n \) equations with \( n \) unknowns \( g(x) = 0 \). We can formulate this as the least squares optimization problem (1.81) \([x^* \text{ solves the system } g(x) = 0 \text{ if and only if it minimizes } \frac{1}{2} \|g(x)\|^2 \text{ and the optimal value is zero}].\)

Here are some other examples: