Exercise 1.1.1.

We have

$$\nabla f(x, y) = \begin{pmatrix} 2x + \beta y + 1 \\ 2y + \beta x + 2 \end{pmatrix}.$$  

Setting $\nabla f(x, y) = 0$, we obtain the system of equations

$$\begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$  

This system has a unique solution (a unique stationary point) except when

$$\beta^2 = 4.$$  

If $\beta^2 = 4$, it can be verified that there is no solution to the above system (no stationary point). Assuming $\beta^2 \neq 4$, for the stationary point to be a local minimum, the Hessian matrix of $f$, which is

$$Q = \begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix},$$  

must be positive semidefinite. But if this is so, $f(x, y)$ will be a convex quadratic function and each local minimum will be global.

The Hessian $Q$ will be positive definite if and only if $\beta^2 < 4$ and positive semidefinite if $\beta^2 = 4$, in which case there is no stationary point by the preceding discussion.
Thus, if $\beta^2 < 4$, there is a unique stationary point which is a global minimum. If $\beta^2 = 4$, there is no stationary point. If $\beta^2 > 4$, there is a unique stationary point which, however, is not a local minimum.

**Exercise 1.1.2.**

(a) We have

\[
\nabla f(x, y) = \begin{pmatrix} 4x^3 - 16x \\ 2y \end{pmatrix}, \quad \nabla^2 f(x, y) = \begin{pmatrix} 12x^2 - 16 & 0 \\ 0 & 2 \end{pmatrix}
\]

so the stationary points of $f$ are $(0, 0), (2, 0)$ and $(-2, 0)$. Now, since

\[
f(2, 0) = f(-2, 0) = 0 \quad \text{and} \quad f(x, y) \geq 0, \quad \forall \ x, y \in \mathbb{R}
\]

we have that $(2, 0)$ and $(-2, 0)$ are global minima of $f$.

Since

\[
\nabla^2 f(0, 0) = \begin{pmatrix} -16 & 0 \\ 0 & 2 \end{pmatrix}
\]

has a positive and a negative eigenvalue, $(0, 0)$ is neither a local maximum nor a local minimum of $f$.

(b) We have

\[
\nabla f(x, y) = \begin{pmatrix} x + \cos y \\ -x \sin y \end{pmatrix}, \quad \nabla^2 f(x, y) = \begin{pmatrix} 1 & -\sin y \\ -\sin y & -x \cos y \end{pmatrix}
\]

Thus the stationary points of $f$ are:

\[
\{((-1)^{(k+1)}, k\pi) \mid k = \text{integer}\}, \quad \{(0, k\pi + \pi/2) \mid k = \text{integer}\}.
\]

Of these, the local minima are

\[
\{((-1)^{(k+1)}, k\pi) \mid k = \text{odd}\}.
\]

(c) We have

\[
\nabla f(x, y) = \begin{pmatrix} \cos x + \cos(x + y) \\ \cos y + \cos(x + y) \end{pmatrix},
\]

\[
\nabla^2 f(x, y) = \begin{pmatrix} -\sin x - \sin(x + y) & -\sin(x + y) \\ -\sin(x + y) & -\sin y - \sin(x + y) \end{pmatrix}
\]
The stationary points are:

$$(\pi, \pi), \left(\frac{\pi}{3}, \frac{\pi}{3}\right) \text{ and } \left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$$

Since

$$\nabla^2 f \left(\frac{\pi}{3}, \frac{\pi}{3}\right) < 0$$

$$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$ is a local maximum.

Regarding $$(\pi, \pi)$$ we have

$$\nabla^2 f(\pi, \pi) = 0.$$ 

To determine if $$(\pi, \pi)$$ is a local minimum or a local maximum, we set

$$x = \pi + \epsilon, \quad y = \pi + \epsilon$$

for $$\epsilon$$ small in absolute value. We have $$f(\pi + \epsilon, \pi + \epsilon) > 0$$ for $$\epsilon < 0$$ and $$f(\pi + \epsilon, \pi + \epsilon) < 0$$ for $$\epsilon > 0$$. Hence $$(\pi, \pi)$$ cannot be a local minimum or a local maximum.

Finally, we have

$$\nabla^2 f \left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) > 0$$

therefore $$\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$$ is a local minimum.

(d) We have

$$\nabla f(x, y) = \begin{pmatrix} -4xy + 4x^3 - 2x \\ 2y - 2x^2 \end{pmatrix} \quad \nabla^2 f(x, y) = \begin{pmatrix} -4y + 12x^2 - 2 & -4x \\ -4x & 2 \end{pmatrix}$$

Hence the only stationary point of $$f$$ is $$(0, 0)$$, where

$$\nabla^2 f(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

Since $$\nabla^2 f(0, 0)$$ has a positive and a negative eigenvalue, $$(0, 0)$$ is neither a local maximum nor a local minimum.

Exercise 1.1.3.
(a) Since the function $f(x^* + \alpha d)$ is minimized at $\alpha = 0$ for all $d \in \mathbb{R}^n$, we have for all $\alpha$ and $i$

$$f(x^* + \alpha e_i) \geq f(x^*),$$

which implies that

$$\lim_{\alpha \to 0^+} \frac{f(x^* + \alpha e_i) - f(x^*)}{\alpha} \geq 0, \quad \lim_{\alpha \to 0^-} \frac{f(x^* + \alpha e_i) - f(x^*)}{\alpha} \leq 0,$$

or

$$\left( \frac{\partial f(x^*)}{\partial x_i} \right) = 0, \quad \forall \ i.$$

(b) Consider the function $f(y, z) = (z - py^2)(z - qy^2)$, where $0 < p < q$ and let $x^* = (0, 0)$.

We first show that $g(\alpha)$ is minimized at $\alpha = 0$ for all $d \in \mathbb{R}^2$. We have

$$g(\alpha) = f(x^* + \alpha d) = f(\alpha d) = (\alpha d_2 - pa^2d_1^2)(\alpha d_2 - q\alpha^2d_1^2) = \alpha^2(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2).$$

Also,

$$g'(\alpha) = 2\alpha(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(-pd_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(d_2 - p\alpha d_1^2)(-qd_1^2).$$

Thus $g'(0) = 0$. Furthermore,

$$g''(\alpha) = 2(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2) + 2\alpha(-pd_1^2)(d_2 - q\alpha d_1^2) + 2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2) + 2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2) + \alpha^2(-pd_1^2)(-qd_1^2).$$

Thus $g''(0) = 2d_2^2$, which is greater than 0 if $d_2 \neq 0$. If $d_2 = 0$, $g(\alpha) = pq\alpha^4d_1^4$, which is clearly minimized at $\alpha = 0$.

Therefore, $(0, 0)$ is a local minimum of $f$ along every line that passes through $(0, 0)$.

Let’s now show that if $p < m < q$, $f(y, my^2) < 0$ if $y \neq 0$ and that $f(y, my^2) \geq 0$ otherwise. Consider a point of the form $(y, my^2)$. We have $f(y, my^2) = y^4(m - p)(m - q)$. Clearly, $f(y, my^2) < 0$ if and only if $p < m < q$ and $y \neq 0$. In any $\epsilon$–neighborhood of $(0, 0)$, there exists a $y \neq 0$ such that for some $m \in (p, q)$, $(y, my^2)$ also belongs to the neighborhood. Since $f(0, 0) = 0$, we see that $(0, 0)$ is not a local minimum.

**Exercise 1.1.5.**
The problem is
\[
\min_{xyz=1} xy + yz + zx.
\]
Let \( z = \frac{1}{xy} \). An equivalent problem is then to minimize
\[
f(x, y) = xy + \frac{1}{x} + \frac{1}{y},
\]
subject to \( x > 0, y > 0 \). To show that \( f \) has a global minimum on \( \mathbb{R}^2_{++} \) = \{ \( x, y \) : \( x > 0, y > 0 \) \} we will show that there is a \( \gamma \) such that the sublevel set
\[
S_{\gamma} = \{(x, y) : f(x, y) \leq \gamma, x > 0, y > 0 \}
\]
is nonempty and compact. Existence of a global minimum of \( f \) would then follow from Proposition A.8 of the textbook. Let \( \gamma = 3 \) (the argument that follows actually works for any \( \gamma \geq 3 \)). The sublevel set \( S_{\gamma} \) is nonempty because it contains \((1, 1)\) [since \( f(1, 1) = 3 \)]. We now show that \( S_{\gamma} \) is compact. Note that any element of \( S_{\gamma} \) must satisfy \( x \geq 1/\gamma \) and \( y \geq 1/\gamma \) [since \( f(x, y) \geq 1/x \) and \( f(x, y) \geq 1/y \)] and \( x \leq \gamma^2, y \leq \gamma^2 \) [since \( f(x, y) \geq xy \geq x/\gamma \) and \( f(x, y) \geq xy \geq y/\gamma \)]. Thus this shows that \( S_{\gamma} \) is bounded, and furthermore that we have
\[
S_{\gamma} = \{(x, y) : \frac{1}{\gamma} \leq x \leq \gamma^2, \frac{1}{\gamma} \leq y \leq \gamma^2, f(x, y) \leq \gamma \}.
\]
Since \( \{(x, y) : \frac{1}{\gamma} \leq x \leq \gamma^2, \frac{1}{\gamma} \leq y \leq \gamma^2 \} \) is closed, and \( f \) is continuous, it follows (using Proposition A.7.(c) from the textbook) that \( S_{\gamma} \) is closed. We have thus shown that \( S_{\gamma} \) is bounded and closed thus it is compact. Thus \( f \) has a global minimum on \( \mathbb{R}^2_{++} \).

The first order necessary condition for optimality is:
\[
y - \frac{1}{x^2} = 0, \quad x - \frac{1}{y^2} = 0.
\]
The only solution to these conditions is \( x^* = 1, y^* = 1 \). Since this is the only critical point, and since we know that \( f \) has a global minimum on \( \mathbb{R}^2_{++} \), then it must be that \( (x^*, y^*) = (1, 1) \) is the global minimum.