Exercise 1.

(a) $K$ is not necessarily polyhedral.

(b) First, note that $K$ is indeed a cone, since $(x, t) \in K \Rightarrow (\lambda x, \lambda t)$ for $\lambda \geq 0$. Next, we will show that it satisfies all four properties of a proper cone:

- **Convex:** Consider $(x_1, t_1), (x_2, t_2) \in K$. We will show that $(\alpha x_1 + (1 - \alpha)x_2, \alpha t_1 + (1 - \alpha)t_2) \in K$ for $\alpha \in [0, 1]$. We have:

  $$||\alpha x_1 + (1 - \alpha)x_2|| \leq \alpha||x_1|| + (1 - \alpha)||x_1|| \leq \alpha t_1 + (1 - \alpha)t_2$$

  where the first inequality follows from the triangle inequality for the Euclidean norm and the second inequality follows since $(x_1, t_1), (x_2, t_2) \in K$. We conclude that $(\alpha x_1 + (1 - \alpha)x_2, \alpha t_1 + (1 - \alpha)t_2) \in K$ and, therefore, the set $K$ is convex.

- **Closed:** An alternative definition for cone $K$ is the following: $K = \{(x, t)|t - ||x|| \geq 0\}$. Define function $f(x, t) = t - ||x||$. Function $f$ is continuous as the sum of two continuous functions. Moreover, let $Y = \{y \in \mathbb{R}^m|y \geq 0\}$. Note that cone $K$ is precisely the inverse image of $Y$, i.e. $\{(x, t)|f(x, t) \in Y\}$. We conclude that $K$ is closed (see Prop. A.7, page 668 of the textbook).

- **Solid:** Consider point $A = (x, t) = (2, \cdots, 2, 100) \in \mathbb{R}^{n+1}$. Obviously, all points that lie in a ball of radius $\epsilon$ around $A$ for $\epsilon < 1$ belong to the cone $K$, therefore the cone is solid.
• Pointed: We need to show that $K \cap (-K) = \{0\}$, i.e. the only point that belongs to the intersection is the origin. Let $(x, t) \in K \cap (-K)$. Then $(x, t)$ satisfies $t \geq ||x||$ and $-t \geq ||-x|| = ||x||$. From those two we conclude that $t = 0$ and $||x|| = 0 \Rightarrow x = 0$.

(c) The dual cone is defined as follows:

$$K^* = \{ (y, s) | (y, s)^T (x, t) \geq 0 \forall (x, t) \in K \}$$

We can write that $(y, s) \in K^*$ satisfies $y^T x + st \geq 0$. From Cauchy-Schwarz we know that

$$|y^T x| \leq ||y|| ||x|| \Rightarrow y^T x \geq -||y|| ||x||$$

So we can write:

$$(y, s) \in K^* \Rightarrow y^T x + st \geq 0 \Rightarrow -||y|| ||x|| + st \geq 0$$

Now note that since $||x|| \leq t$ ($(x, t) \in K$) and $t \geq 0$ we get that $||y|| \leq s$, i.e. $K^* = K$. We conclude that the second-order cone is self dual.

Exercise 2.

(a) Let $z_1, z_2 \in S + T$. We want to show that $\alpha z_1 + (1 - \alpha) z_2 \in S + T$ for $\alpha \in [0, 1]$. First, note that we can write $z_1 = s_1 + t_1$ and $z_2 = s_2 + t_2$, where $s_1, s_2 \in S$ and $t_1, t_2 \in T$, since $z_1, z_2 \in S + T$. Now

$$z_3 = \alpha z_1 + (1 - \alpha) z_2 = \alpha (s_1 + t_1) + (1 - \alpha) (s_2 + t_2) = [\alpha s_1 + (1 - \alpha) s_2] + [\alpha t_1 + (1 - \alpha) t_2] = s_3 + t_3$$

where $s_3, t_3$ belong to $S, T$ resp. since $S, T$ are convex and, thus, $z_3 \in S + T$. We conclude that $S + T$ is convex.

(b) If $S, T$ are closed and convex then $S + T$ is not necessarily closed. Consider the following counterexample:

$$S = \{(x, y) | xy \geq 1, x \geq 0, y \geq 0\} \text{ and } T = \{(x, y) | x \in \mathbb{R}, y = 0\}$$

It is straightforward to see that both $S, T$ are closed and convex. Furthermore,

$$S + T = \{(x, y) | x \in \mathbb{R}, y > 0\}$$
which is obviously not closed.

- However if $S$ is closed and $T$ is compact, i.e. closed and bounded, then $S + T$ is closed. To show that this true, consider an infinite sequence of points $z_k \in S + T$, which converges to some limit point $\bar{z}$. We have to show that $\bar{z} \in S + T$. First, note that since points in $\{z^k\}$ belong to $S + T$ they can be written in the form $z_k = s_k + t_k$, where $s_k, t_k$ belong to $S, T$ respectively. In this way we define two sequences $\{s^k\}$ and $\{t^k\}$ of points $\in S, T$ resp., which satisfy that $z_k = s_k + t_k$.

Also note that $T$ is compact, therefore for every infinite sequence of points in $T$ there exists an (infinite) subsequence which converges to a limit point $\bar{t}$. We can restrict our attention to this subsequence, i.e. consider the $z_k$'s, $s_k$'s and $t_k$'s that correspond to this subsequence and define the sequences $\{z^{k'}\}, \{s^{k'}\}, \{t^{k'}\}$. Now note that the following holds for those sequences $s_{k'} = z_{k'} - t_{k'}$. Furthermore, the right hand side converges to some limit point from continuity and since $\{z^{k'}\}$ converges (as an infinite subsequence of a convergent sequence) and $\{t^{k'}\}$ converges (we consider a convergent subsequence of a sequence of points in a compact set). Moreover the limit point is equal to $\bar{z} - \bar{t}$. However, set $S$ is closed and therefore the limit point of a sequence of points that belong to the set also belongs to the set. We conclude that $\bar{s} = \bar{z} - \bar{t} \in S$ and $\bar{z} = \bar{s} + \bar{t} \in S + T$ since $\bar{s} \in S$ and $\bar{t} \in T$.

Exercise 3.

(a) We will show that $f$ is convex using the standard definition of convexity, i.e. a function $f$ is convex if and only if for every $z_1, z_2 \in \text{dom}(f)$ and every $\alpha \in [0, 1]$

$$f(\alpha z_1 + (1 - \alpha)z_2) \leq \alpha f(z_1) + (1 - \alpha)f(z_2)$$
We have
\[
f(\alpha x_1 + (1 - \alpha)x_2, \alpha t_1 + (1 - \alpha)t_2) = [\alpha t_1 + (1 - \alpha)t_2]g(\frac{\alpha x_1 + (1-\alpha)x_2}{\alpha t_1 + (1-\alpha)t_2})
\]
\[
= [\alpha t_1 + (1 - \alpha)t_2]\left( \frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} x_1 + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} x_2 \right)
\]
\[
\leq [\alpha t_1 + (1 - \alpha)t_2]\left( \frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} g(x_1) + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} g(x_2) \right)
\]
\[
= \alpha f(x_1, t_1) + (1 - \alpha) f(x_2, t_2)
\]

We conclude that \( f(x, t) \) is indeed convex in its domain, i.e. \( \mathbb{R}^n \times \mathbb{R}_{++} \).

(b) We will show that \( f \) is convex using the following definition of convexity: a function \( f \) is convex if it is convex when restricted to any line that intersects its domain, i.e. \( f \) is convex if and only if for all \( x \in \text{dom} f \) and all \( v \), the function \( g(t) = f(x + tv) \) is convex (on its domain, \( \{ t \mid x + tv \in \text{dom} f \} \)).

Consider an arbitrary line that intersects the domain of \( f \), i.e. \( X = Z + tV \), where \( Z \succ 0 \) and \( Z + tV \succ 0 \). We have
\[
g(t) = -\log \det (Z + tV)
\]
\[
= -\log \det (Z^{1/2}(I + tZ^{-1/2}VZ^{-1/2})Z^{1/2})
\]
\[
= -\sum_{i=1}^{n} \log (1 + t\lambda_i) - \log \det Z
\]
where the \( \lambda_i \)'s are the eigenvalues of \( Z^{-1/2}VZ^{-1/2} \). Therefore we have
\[
g'(t) = -\sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i} \quad \text{and} \quad g''(t) = \sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + t\lambda_i)^2}
\]
Since \( g''(t) \geq 0 \) we conclude that \( f \) is convex.

(c) First, note that since \( u, v \) are strictly positive we can write
\[
f(u, v) = \sum_{i=1}^{n} u_i \ln(\frac{u_i}{v_i}) = \sum_{i=1}^{n} u_i (-\ln(\frac{v_i}{u_i}))
\]
Now note that function \( g(x) = -\ln x \) is convex, so we can use the result of part (a) to conclude that \( f(u, v) \) is convex as the sum of convex functions.

Exercise 5.1.2.

The problem is
\[
\min f(x) = 10x_1 + 3x_2
\]
subject to $5x_1 + x_2 \geq 4$, $x_1, x_2 = 0$ or $1$.

(a) See Figure 1.

(b) The Lagrangian function is

$$L(x, \mu) = 10x_1 + 3x_2 + \mu(4 - 5x_1 - x_2)$$

and the dual function is

$$q(\mu) = \inf_{x_1, x_2 \in (0, 1)} L(x, \mu) = 4\mu + (10 - 5\mu)x_1 + (3 - \mu)x_2$$

\[
\begin{cases} 
4\mu & \text{if } \mu \leq 2, \\
10 - \mu & \text{if } 2 \leq \mu \leq 3, \\
13 - 2\mu & \text{if } \mu \geq 3,
\end{cases}
\]

(see Figure 2).

(c) From (a), we see that $x^* = (1, 0)$ and $f^* = 10$. From (b), we see that $q^* = 8$. Thus there is a duality gap of $f^* - q^* = 2$ and there is no Lagrange multiplier.

Exercise 5.1.3.
A straightforward calculation yields the dual function as

\[ q(\lambda) = \min_x \{ \|z - x\|^2 + \lambda'Ax \} = -\frac{\|A'\lambda\|^2}{4} + \lambda'Az. \]

Thus the dual problem is equivalent to

\[ \min_{\lambda} \left\{ \frac{\|A'\lambda\|^2}{4} - \lambda'Az + \|z\|^2 \right\} \]

or

\[ \min_{\lambda} \left\| z - \frac{A'\lambda}{2} \right\|^2. \]

This is the problem of projecting \( z \) on the subspace spanned by the rows of \( A \).

Exercise 5.2.2.

Without loss of generality, we may assume that there are no equality constraints, so that the problem is

\[
\text{minimize} \quad f(x)
\]

subject to \( x \in X, \quad a'_j x - b_j \leq 0, \quad j = 1, \ldots, r. \)

Let \( X = C \cap P \), and let the polyhedron \( P \) be described in terms of linear inequalities as

\[ P = \{ x \in \mathbb{R}^n \mid a'_j x - b_j \leq 0, \quad j = r + 1, \ldots, p \}, \]
where \( p \) is an integer with \( p > r \). By applying Lemma 5.2.2 with
\[
S = \{ x \in \mathbb{R}^n \mid a'_j x - b_j \leq 0, \ j = 1, \ldots, p \},
\]
and \( F(x) = f(x) - f^* \), we have that there exist scalars \( \mu_i \geq 0, \ j = 1, \ldots, p \), such that
\[
f^* \leq f(x) + \sum_{j=1}^{p} \mu_j (a'_j x - b_j), \quad \forall \ x \in C.
\]
For any \( x \in X \) we have \( \mu_j (a'_j x - b_j) \leq 0 \) for all \( j = r + 1, \ldots, p \), so the above relation implies that
\[
f^* \leq f(x) + \sum_{j=1}^{r} \mu_j (a'_j x - b_j), \quad \forall \ x \in X,
\]
or equivalently
\[
f^* \leq \inf_{x \in X} \{ f(x) + \sum_{j=1}^{r} \mu_j (a'_j x - b_j) \} = q(\mu) \leq q^*.
\]
By using the weak duality theorem (Prop. 5.1.3), it follows that \( \mu \) is a Lagrange multiplier and that there is no duality gap.

In Example 5.2.1, we can set \( C = \{ x \in \mathbb{R}^2 \mid x \geq 0 \} \) and \( P = \{ x \in \mathbb{R}^2 \mid x_1 \geq 0 \} \). Then evidently \( X = C \) and \( f \) is convex over \( C \). However, \( ri(C) = int(C) = \{ x \in \mathbb{R}^2 \mid x > 0 \} \), while every feasible point \( x \) must have \( x_1 = 0 \). Hence no feasible point belongs to the relative interior of \( C \), and as seen in Example 5.2.1, there is a duality gap.

**Exercise 5.3.1.**

Assume that there exists an \( \bar{x} \in X \) such that \( g_j(\bar{x}) < 0 \) for all \( j \). By Prop. 5.3.1, the set of Lagrange multipliers is nonempty. Let \( \mu \) be any Lagrange multiplier. By assumption, \( -\infty < f^* \), and we have
\[
-\infty < f^* \leq L(\bar{x}, \mu) = f(\bar{x}) + \sum_{j=1}^{r} \mu_j g_j(\bar{x}),
\]
or
\[
-\sum_{j=1}^{r} \mu_j g_j(\bar{x}) \leq f(\bar{x}) - f^*.
\]
We have
\[
\min_{i=1, \ldots, r} \{-g_i(\bar{x})\} \leq -g_j(\bar{x}), \quad \forall \ j,
\]
so by combining the last two relations, we obtain

$$
\left( \sum_{j=1}^{r} \mu_j \right) \min_{i=1,\ldots,r} \left\{ -g_i(\bar{x}) \right\} \leq f(\bar{x}) - f^*.
$$

Since $\bar{x}$ satisfies $g_j(\bar{x}) < 0$ for all $j$, we have

$$
\sum_{j=1}^{r} \mu_j \leq \frac{f(\bar{x}) - f^*}{\min_{j=1,\ldots,r} \left\{ -g_j(\bar{x}) \right\}}.
$$

Hence the set of Lagrange multipliers is bounded.

Conversely, let the set of Lagrange multipliers be nonempty and bounded. Consider the set

$$
B = \{ z \mid \text{there exists } x \in X \text{ such that } g(x) \leq z \}.
$$

Assume, to arrive at a contradiction, that there is no $\bar{x} \in X$ such that $g(\bar{x}) < 0$. Then the origin is not an interior point of $B$, and similar to the proof of Prop. 5.3.1, we can show that $B$ is convex, and that there exists a hyperplane whose normal $\gamma$ satisfies $\gamma \neq 0$, $\gamma \geq 0$, and

$$
\gamma' g(x) \geq 0, \quad \forall x \in X.
$$

Let now $\mu$ be a Lagrange multiplier. Using Eq. (1), we have for all $\beta \geq 0$

$$
f^* = \inf_{x \in X} L(x, \mu) \leq \inf_{x \in X} L(x, \mu + \beta \gamma) \leq \inf_{x \in X, g(x) \leq 0} L(x, \mu + \beta \gamma) \leq \inf_{x \in X, g(x) \leq 0} f(x) = f^*,
$$

where the last inequality holds because $\mu + \beta \gamma \geq 0$, and hence $(\mu + \beta \gamma)' g(x) \leq 0$ if $g(x) \leq 0$. Hence, equality holds throughout in the above relation, so $\mu + \beta \gamma$ is a Lagrange multiplier for all $\beta \geq 0$. Since $\gamma \neq 0$, it follows that the set of Lagrange multipliers is unbounded – a contradiction.

**Exercise 5.3.3.**

For simplicity and without loss of generality, assume that $A(x^*) = \{1, \ldots, r\}$, and denote

$$
h_j(x) = \nabla g_j(x^*)'(x - x^*), \quad \forall j.
$$

By Prop. 5.1.1, $\mu \in M^*$ if and only if $x^*$ is a global minimum of the convex problem

$$
\begin{align*}
\text{minimize} \quad & \nabla f(x^*)'(x - x^*) \\
\text{subject to} \quad & x \in X, \quad h_j(x) \leq 0, \quad j = 1, \ldots, r,
\end{align*}
$$

(1)
while \( \mu \) is a Lagrange multiplier. The feasible directions of \( X \) at \( x^* \) are the vectors of the form \( d = x - x^* \) where \( x \in X \). Hence the assumption that there exists a feasible direction \( d \) with the property described is equivalent to the existence of an \( \bar{x} \in X \) such that \( h_j(\bar{x}) < 0 \) for all \( j \).

If there exists a feasible direction \( d \) with \( \nabla g_j(x^*)'d < 0 \) for all \( j \), then by Prop. 3.3.12, the set \( M^* \) is nonempty. Applying the result of Exercise 5.3.1 to problem (1), we see that the set \( M^* \) is bounded. Conversely, if \( M^* \) is nonempty and bounded, again applying the result of Exercise 5.3.1, we see that there exists \( \bar{x} \in X \) such that \( h_j(\bar{x}) < 0 \) for all \( j \), and hence also there exists a feasible direction with the required property.