Combinatorial Geometry

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Previous Lecture

• Algorithm for matching $A$ in $B$:
  – Take any pair $a,a' \in A$, let $r = ||a-a'||$
  – Find all pairs $b,b' \in B$ such that $||b-b'||=r$
  – For all such pairs
    • Compute $t$ that transforms $(a,a')$ into $(b,b')$
    • Check if $t(A) \subseteq B$
Combinatorial Question

• Given a set $A$ of $n$ points in the plane, what is the maximum number of $p, p' \in A$ such that $||p-p'||=1$?
  – Erdős’46: $O(n^{3/2})$
  – Jozsa, Szemerédi’73: $o(n^{3/2})$
  – Beck, Spencer’84: $O(n^{1.44...})$
  – Spencer, Szemeredi, Trotter’84: $O(n^{4/3})$
  – Szekely’96: $O(n^{4/3})$, proof in 4 slides
Crossing Number

• Crossing number of a graph $G$: smallest $k$ such that $G$ can be drawn on the plane with at most $k$ edges crossing

• Interested in bounds of the form $k \geq f(n, e)$
Simple Bound

• We know that $k \geq e - 3n$

• Proof:
  – Assume $G$ with $k < e - 3n$
  – Then there is a graph with $k - 1$ crossings and $e - 1$ edges
  – ..... 
  – There is a graph with 0 crossings and $e - k$ edges
  – But $e - k \leq 3n$ – a contradiction (from an upper bound on the number of edges in a planar graph)
Bounds

- The earlier lower bound is pretty weak. E.g., it lower bounds $k$ by at most $e$
- Complete graph has crossing number $\Omega(n^4)$
- Need to “amplify” the bound
Probabilistic Amplification

• Given $G$, construct $G'$ by random sampling
• Each node is included in $G'$ with probability $p = \frac{4}{n/e}$ (assume $e \geq 4n$)
• The expected parameters $n', e', k'$ of $G'$ are:
  – $E[n'] = pn$
  – $E[e'] = p^2e$
  – $E[k'] \leq p^4k$
Proof

• The “weak bound”: $k' + 3n' - e' \geq 0$
• Thus $E[k' + 3n' - e'] = E[k'] + 3E[n'] - E[e'] \geq 0$
• We get:

$$p^4k + 3pn - p^2e \geq 0$$
$$p^3k \geq pe - 3n = 4n - 3n = n$$
$$k \geq e^3/(4^3 n^2) = \Omega(e^3/n^2)$$
$$e = O\left( (kn^2)^{1/3} \right) \ [Leighton'83]$$
$$[Ajtai, Chvatal, Newborn, Szemeredi'82]$$
Number of Unit Distances

- Nodes = points = $n$
- Multi-edges defined by arcs $\geq$ #unit distances
- Keep one out of $\leq 4$ edges, so we get a graph
- # crossings $\leq 2n^2$
- $e = O((kn^2)^{1/3}) \rightarrow$ #unit distances $= O((n^4)^{1/3})$
Other Bounds

- Given $n$ points, what is the number of distinct distances between them?
  - $\Omega(n^{1/2})$ Paul Erdős’ 1946
  - $\Omega(n^{2/3})$ Leo Moser, 1952,
  - $\Omega(n^{5/7})$ (Fan Chung, 1984),
  - $\Omega(n^{4/5}/\log n)$ (Fan Chung, Endre Szemerédi, W. T. Trotter, 1992),
  - $\Omega(n^{4/5})$ (László Székely, 1993),
  - $\Omega(n^{6/7})$ (József Solymosi, C. D. Tóth, 2001),
  - $\Omega(n^{(4e/(5e - 1)) - e})$ (Gábor Tardos, 2003),
  - $\Omega(n^{((48 - 14e)/(55 - 16e)) - e})$ (Nets Hawk Katz, Gábor Tardos, 2004).
  - $\Omega(n^{1 - o(1)})$ (Larry Guth and Nets Hawk Katz, 2010)
Improved Algorithm

• Take the pair $a, a' \in A$ with the lowest multiplicity in $B$; let $r=||a-a'||$

• Find all pairs $b, b' \in B$ such that $||b-b'||=r$

• For all such pairs
  – Compute $t$ that transforms $(a, a')$ into $(b, b')$
  – Check if $t(A) \subseteq B$
Analysis

• For a distance $t$, let $m_A(t)$ be the multiplicity of $t$ in $A$
• $\sum_t m_B(t) \leq n^2$
• There are at least $n^{1-o(1)}$ different $t$’s such that $m_A(t) \geq 1$
• So, if there is a match, there must exist $t$ such that $m_A(t) \geq 1$ and $m_B(t) \leq n^{1+o(1)}$
• Algorithm has running time $O(n^{2+o(1)})$
Higher Dimensions

• What is the number of unit distances between \(n\) points in \(\mathbb{R}^4\)?

• At least \(n^2/4\):
  – Let \(A=\{(x,y,z,u): x^2+y^2=1, \ z=u=0\}\)
  – Let \(B=\{(x,y,z,u): z^2+u^2=1, \ x=y=0\}\)
  – For any \(a \in A, \ b \in B\), we have \(||a-b||^2=2\)
  – Take \(n/2\) points from \(A\) and \(n/2\) points from \(B\)