Chapter 1: Hints for holography

1.1 Prelude: Gravity vs. Other Interactions

Which of these interactions does not belong?

a) Electromagnetism
b) Weak interaction (nuclear)
c) Strong interactions (nuclear)
d) Gravity

Why?

(a) - (c) are all described by gauge theories defined on a fixed spacetime.

E.g., (a) gauge field + Dirac equation

⇒ QED
Electroweak: $SU(2) \times U(1)$

Strong: $SU(3)_c$

The basic theoretical structure is well-understood:

path integral $+ \text{Wilsomian RG}$

$\Rightarrow$ any calculation can be reduced to algorithm

(e.g. lattice)

but of course many, many interesting questions remain.

Gravity

Classical gravity $= \text{spacetime (GR)}$

Quantum gravity: far from being understood

- Spacetime should become dynamical, what replaces it?
- Quantum nature of black holes
- Spacetime singularities
- Initial conditions of the Universe (?)
The remarkable discovery: (Maldacena 1997)

\texttt{AdS/CFT, gauge/gravity duality, holographic duality)}

gauge theory $\iff$ quantum gravity

The same theory, expressed in different variables, exposes deep aspects of both theories.

Goal of the course:

- Motivate and derive the relation
- Work out the (partial) dictionary
- \texttt{AdS gravity $\Rightarrow$ difficult questions in gauge theories, many-body physics}
- gauge theories $\iff$ quantum gravity
- Outline open questions
I just checked this morning, Maldacena's original paper has 7264 citations on SLAC database.

But in my opinion, the subject is still in its infancy.

As we will see during the course, there are many elementary issues which are not understood.

What have been found so far likely just scratch the surface.

(e.g., so far only QG in asymptotic AdS spacetime)

Looking at things in perspective, 0, when fully understood, should be up there comparable to other milestones in physics:

Newton's discovery of universal gravitation, Maxwell,
Newton's discovery of universal gravity

Maxwell

Boltzmann

Einstein

I hope to convey both excited and mysteries! For myself, teaching this course is an exploration by itself, an adventure, or opportunity to indulge.

The outline only reflects what I have in mind right now. As we go along, I may change mind and deviate. Also, I have no idea at the moment how much I could cover.

I will make my best effort to make things self-contained, even though you may imagine there

...
will be many things which I will only quote (or motivate) rather than derive.

Your feedback will be crucial!

Some logistics:

1) No text books, many reviews, very suitable as a whole. I will give references along the way.

2) Recitutions (? ) time (? )

3) Office hours: Tuesday, 8:30 - 9:30 pm

No TA, advanced GAS are excellent resource!

4) Grades are based on tests and partially on your own grade (experiment!?)
Posts: every two weeks. Very important, help you understand. Feedback important!

**Relation: Emergence of Gravity:**

From gauge theory point of view, it implies (quantum) gravity + spacetime can emerge from a non-gravitational theory.

The idea itself is not new.

E.g., A. Sakharov already proved 1967 mathematically who observed certain CPT phenomena have descriptions similar to those in GR.

Indeed, it appears natural to imagine gravitons (spin-2 massless particle) may
arise as bound states of spin-1 (photons, gluons) or fermions (spin-\(\frac{1}{2}\)) particles.

However, there exists a Weinberg-Witten theorem is very powerful, not but without interesting loop holes.
However, there exists a famous theorem by Weinberg-Witten which says this is impossible. 

PLB 96 059 (1980).

Theorem 1: A theory that allows the construction of a Lorentz-covariant conserved 4-vector current $J^a$ cannot contain massless particles of spin $j > \frac{1}{2}$ with nonvanishing values of the conserved charge $\int J^a d^3x$.

Th. 2: A theory that allows a conserved Lorentz-covariant stress tensor $T^{\mu\nu}$, which $\int T^{\mu\nu} d^3x$ is the energy-momentum 4-vector cannot contain massless particles of spin $j > 1$. 

These theorems turn out to be surprisingly simple to prove, follow only from simple kinematics (as you might be able to imagine).

Proof: Suppose there exist massless particles of spin $-\frac{1}{2}$, one-particle states: $|k^\mu, \sigma\rangle$

\[ k^\mu = (k^0, \vec{k}) \quad \sigma = \pm \frac{1}{2} \]

(consult Weinberg, vol. I, sec. 2.5)

\[ \hat{R}(\theta, \hat{k}) | k^\mu, \sigma\rangle = e^{i \sigma \theta} | k^\mu, \sigma\rangle \]

Rotational operator by $\theta$ around $\hat{k}$ direction.

Now consider conserved current $J^\mu$ with

\[ Q = \int d^3 x \quad J^0 \]

\[ Q | k^\mu, \sigma\rangle = \sigma | k^\mu, \sigma\rangle \]
The conserved, covariant stress tensor

\[ \Theta^\mu_{\nu} \ e P^\mu = \int d^4x \ T^{\mu\nu} \]

\[ P^\mu \mid k^\mu, 0 \rangle = k^\mu \mid k^\mu, 0 \rangle \]

want to show that:

if \( q \neq 0 \), \( j \leq \frac{1}{2} \)

if \( Tw \) exists, \( j \leq 1 \)

(i) Lorentz invariance \( \Rightarrow \) (post)

\[ \langle k^\mu, 0 \mid J^\mu \mid k'^\mu, 0 \rangle \xrightarrow{k' \rightarrow k} \frac{q \ k^\mu}{k^0} \frac{1}{(2\pi)^3} \]

\[ \langle k^\mu, 0 \mid Tw \mid k'^\mu, 0 \rangle \xrightarrow{k' \rightarrow k} \frac{kk'k'\mu}{k^0} \frac{1}{(2\pi)^3} \]

\( k^0 \) downstairs: related to behavior of states under boost

\[ \langle k^\mu, 0 \rangle \mid k'^\mu, 0 \rangle = \delta_{0,0'} \delta^{(3)}(k-k') \]

\( k^0 \) downstairs: related to normalization of states under boost.
(ii) Note: \( R^2 = 0, \quad R'^2 = 0 \)

\[
\mathbf{k} \cdot \mathbf{k'} < 0
\]

\( \mathbf{k} + \mathbf{k'} \) is time-like.

Choose a frame \( \mathbf{k} + \mathbf{k'} = 0 \)

\[
\mathbf{k} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{k'} = \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix}
\]

(iii) Under a rotation by \( \theta \) around \( s \)-direction:

\[
\hat{R}(\theta) | k^0, 0 > = e^{i00} | k^0, 0 >
\]

\[
\hat{R}(\theta) | k'^0, 0 > = e^{-i00} | k'^0, 0 >
\]

\[
\langle k'^0, j | \hat{R}(\theta) J^m \hat{R}(\theta) | k^0, j >
\]

\[
\Rightarrow \begin{pmatrix} 0 \end{pmatrix} e^{2i\theta j} \langle k^0, j | J^m | k^0, j >
\]

\[
= \Lambda^m_n \langle k^0, j | J^m | k^0, j >
\]

\( \Lambda^m_n \) : Rotational matrix corresponding to \( \hat{R}(\theta) \) on a vector.
\[ \text{Similarly} \quad e^{2\pi j \theta} < k^0, j | T^{k_0} | k^0, j > = 0 \wedge \kappa \wedge \lambda < k^0, j | T^\lambda | k^0, j > \]

Note: \( \Lambda^{k_0} \) only has eigenvalues: \( e^{\pm i \theta}, 1 \)

Thus \( < k^0, \hat{\sigma} | J^k | k, \hat{\sigma} > \) can only be

nonzero if \( j \leq \frac{1}{2} \). \(( j > \frac{1}{2} \text{ contradicts with } 3) \)

\( < k^1, 0 | T^{k_0} | k, 0 > \) can only be

nonzero if \( j \leq 1 \). \(( j > 1 \text{ contradicts with } 3) \)

Q.E.D.
Remarks:  
1) The theorems apply to both "elementary" and "composite" particles.

2) The theorems do not forbid photons, because they carry no charge.

Consider $SU(2)$ $X-M$.

$$A_{\mu}^a, \ a=1, 2, 3$$

$$\Rightarrow \ A_3^\mu, \ A_\mu^+ = \frac{1}{2} (A_\mu^1 \pm i A_\mu^2)$$
A massless particles charged under U(1) subgroup \( \hat{\text{Spin}}-1 \) generated by \( \frac{\sigma^3}{2} \).

But there does not exist a conserved, Lorentz-covariant \( J^\mu \) corresponding to such a U(1). (p07 problem)

3) * It does not forbid graviton from GR.

As in GR, there is no conserved, Lorentz covariant stress tensor.

4) The theorem does apply to all renormalizable QFTs including non-Abelian gauge theories.

\[ \Rightarrow \) no emergent gravitons \( \Rightarrow \) no emergent gravitons \( \Rightarrow \) no emergent gravitons \( \Rightarrow \) no emergent gravitons \( \Rightarrow \) no emergent gravitons.

E.g. QCD does have spin-2 excitations, but all massive (and higher spin).
The W-W theorem does make a few assumptions:

1) Lorentz invariance

2) Gravity decouples in the same spacetime as the original QFTs.

As we will see, it is precisely this second seemingly obvious assumption that gauge/gravity duality takes advantage of.

The emergent gravity (or spacetime) does not move in the same space as gauge fields!

Before presenting the duality, we first discuss various important clues.
Last lecture: we ended with the W-W theorem which forbids massless spin-2 particle in the same spacetime as a QFT lives.

Loophole: graviton can live in a different, one-dimensional higher spacetime.

New couple of lectures: hints of holographic duality from gravity side, BTs.

But for historical reasons:

1) Both important elements of entwining,
2) not just be other hints.
1.2. Black hole thermodynamics

1.2.1 Significance of Planck scale

Fundamental constants in nature:

\[
\hbar, \quad G_N, \quad c = 1
\]

\[
\ell_P = \sqrt{\hbar G_N} \approx 10^{-33} \text{ cm}
\]

\[

m_P = \sqrt{\frac{\hbar}{G_N}} \approx 10^{19} \text{ GeV}
\]

(Planck: (1899): units...that would, independently of special bodies and substances, necessarily retain their significance for all times and all cultures, even extraterrestrial and extra-human ones, ...

- Strength of an interaction:

\[
E_M = \frac{e^2}{r}, \quad \text{minimal distance} \quad \theta = \frac{\hbar}{m c} \quad u, v, c
\]

Consider some fundamental particle of mass.
\[ \lambda_e \sim \frac{\nu^{(re)}}{m} \sim \frac{e^2}{\hbar} = 2 \]

For QED, \( \lambda < 1 \).

Note that we use \( \lambda \) for QED greater than 1, then \( \lambda_m \neq \lambda_c \), instead

\[ \frac{V(r_m)}{m} \ll 1 \Rightarrow \frac{r_m}{m} \frac{e^2}{m} \ll \frac{\hbar}{m} \quad (*) \]

we will see such examples later on.

\[ \text{Gravity:} \quad V = \frac{G_N m^2}{r} \]

Again consider the potential energy at \( r_c \).

\[ \lambda_g \sim \frac{\nu^{(re)}}{m} \sim \frac{G_N m^2}{\frac{\hbar}{m}} \ll \frac{G_N}{\frac{\hbar}{m}} m^2 \ll \left( \frac{m}{m_p} \right)^2 \]

If \( m \ll m_p \), then \( \lambda_g < 1 \)

E.g., for \( e^- \), \( m_e \sim 5 \times 10^{-4} \text{ GeV} \) \( \Rightarrow \frac{\lambda_g}{\lambda_e} \sim 10^{-43} \)

(gravitational very weak!)
For $M \gg m_p$, $\lambda g \gg 1$, again introducing $r_m$ so that

$$V(r_m) \sim O(1) \Rightarrow \frac{G r_m}{r_m} \sim O(1)$$

$$\Rightarrow r_m \sim G r_m \gg r_c \quad (***)$$

Note:
1) $r_m$ in (***) is an intrinsic classical quantity.

It gives the length scale where classical gravity becomes strong for a given $M$.

Since $r_m \gg r_c$, classical gravity suffices at such a scale. (Need GTR)

This is in contrast with $r_m$ in (**) which is an intrinsic QM quantity.

$$e = e_f h \quad e_f = \text{classical charge for field}$$

Similarly: How effective QM couple for gravity

$e_f$ (counterpart of $G r_m$)
(2) Being proportional to \( m \), \( F = \frac{G m_1 m_2}{r^2} \) can be arbitrarily large provided \( m_2 \) is large enough.

(3) While we have been talking about point particles, the conclusion trivially applies to finite-sized objects (using center of mass energy) with size sufficiently small.

What happens in GR at this radius for an object (with size smaller than \( r_{\text{in}} \))?
a black hole with \( R_s = \frac{2Gm}{c^2} \) !

\[ R_s \text{: minimal length which we can probe a BH} \]

To summarize for gravity

If \( m \leq m_p \), gravity is weak, QG effects small.

(Or center of mass energy)

If \( m \gg m_p \), classical gravity takes over.

(Or center of mass energy)

Only for \( m \gg m_p \) or \( E \gg m_p \), quantum gravity important.

Surprising consequences:

\[ e^- e^- \rightarrow E \]

\[ E \text{ at QFT: } \text{prob} \ e \sim \frac{1}{E}, \quad \text{short-distance} \]

\[ \text{Incl. gravity: when } E \gg m_p, \quad \text{prob} \ e \sim R_s \sim E \]

\[ \text{for short-distance}, \quad \text{prob} \rightarrow 0 \]
$E_p$ is the minimal localization length!

(Just argument of next page.)

If this were the whole story, life would be much simpler and also much less exciting, we could let our grand, grand --- our kids worry about QG, as in no foreseeable future we could reach $E_p$, (ii) physics at one scale is measure $E_p$ compared to all other physics!

Of course, life is much more interesting than dimensional analysis and no-go theorems.

Remarkable thing: BH can make QG effects manifest at macroscopic level!
An equivalent argument: $\ell_p$ minimal localization length.

Consider uncertainty in position: $\Delta x$

$$\Delta x > \frac{h}{\Delta x} \Rightarrow \Delta x > \frac{h}{\ell_p}$$

$$\Rightarrow \Delta x > \ell_p$$
Various regimes: (fixed energy scales we are interested in)

- Classical gravity: $\hbar \to 0$, $G_N$ finite, $\hbar G_N \to 0$
- QFT: $\hbar$ finite, $G_N \to 0, \hbar G_N \to 0$
  no gravity
- QFT in a curved, $\hbar$ finite, $G_N \to 0$
  space time

\[ G_N M \text{ remains finite } \Rightarrow \text{ curved space time} \]
\[ (G_N \to 0, \hbar \to 0) \]
\[ G_N \text{ The (QFT) } < G_N M \]

no back reaction from QFT matter.

(semi-classical regime for QG)

- QG: $G_N, \hbar$ finite
Real-life black holes form from gravitational collapse. The full solution is somewhat complicated.

Here we consider an idealized situation, with time reversal symmetry.
1.2.2 Black hole geometry

The metric from an object of mass $M$ (spherically symmetric, non-rotation, neutral):

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2$$

\( f = 1 - \frac{2GM}{r} = 1 - \frac{r_s}{r} \)

$r = r_s = 2GM$: Event horizon

Some simple facts: (all can be shown directly from \( f \), some also)

1) The spacetime is non-singular at the horizon, coordinate singularity.

2) $r = r_s$ surface is a null hypersurface.

For $r = \text{conv}$

$$ds^2 = -f dt^2 + r^2 dv^2$$

\( \Rightarrow \) $F \phi = \int F r^2 \to 0 \quad r \to r_s$
3) The horizon acts as a surface of infinite redshift.

\[ \text{Consider an observer at a hypersurface } r = r_h = r_s \]

and an observer at \( r = \infty \)

At \( r = \infty \),

\[ \text{d}S^2 \rightarrow -\text{d}t^2 + \text{d}r^2 + r^2 \text{d}\Omega^2 \]

\( t \) is the proper time for the observer at infinity.

(Schwarzschild time)

The proper time for an observer near the horizon is:

\[ \text{d}T_h = f^{\frac{1}{2}} \text{d}t = \left(1 - \frac{r_s}{r_h}\right)^{\frac{1}{2}} \text{d}t \]

Thus \( \frac{\text{d}T_h}{\text{d}t} \rightarrow 0 \) as \( r_h \rightarrow r_s \)

If \( O_h \) sends a signal of frequency \( \omega_h \),

the frequency observed by the observer \( O_r \) is then

\[ \omega_r = \omega_h \left(1 - \frac{r_h}{r_s}\right)^{\frac{1}{2}} \]

as \( r_h \rightarrow r_s \)
4) It takes a free-fall traveler finite proper time to reach the horizon, but infinite Schwarzschild time.

5) Once inside the horizon, the traveler cannot send signal to outside friends, nor can he escape himself. Note: \( \text{if changes sign, } r \text{ becomes "time".} \)

6) \( r=0 \) is a curvature singularity. Curvature invariants blow up there.

7) Two intrinsic quantities about the horizon:

(a) Geometric area of a spatial section of the event horizon

\[
A = 4\pi r_h^2 = 4\pi \left( \frac{2}{G_N M} \right)^2
\]

\[
= 16\pi G_N^2 M^2
\]
(b) Surface gravity.

Acceleration of a stationary observer at the horizon as measured by an observer at infinity. (See e.g. Wald, p. 158 and Sec.12.5)

\[ k = \frac{1}{2} f'(r_s) = \frac{1}{2r_s} = \frac{1}{4G\text{HM}} \]

End of 7-2

Killing spacetime and causal structure

Consider the region near the horizon. Introduce proper distance \( \frac{r-r_s}{r_s} \ll 1 \) from the horizon:

\[ ds^2 = \frac{dr^2}{f} \rightarrow ds^2 = \frac{dr}{f^{1/2}} = \frac{dr}{(f'(r_s)^2)(r-r_s)^{1/2}} \]

\[ \rho = 2 \int r_s \sqrt{r-r_s} \quad (\text{for } r \geq r_s) \]

\[ f = \frac{r-r_s}{r_s} = \frac{\rho^2}{4r_s^2} \]

\[ = f'(r_s) (r-r_s) = \frac{1}{4} (f'(r_s))^2 \rho^2 = k^2 \rho^2 \]
\[ ds^2 = \frac{r^2}{4r_s^2} \, dt^2 + d\rho^2 + r_s^2 \, d\Omega^2 \]  \hspace{1cm} (\text{for } s^2 = \ell^2 + \rho^2)

\[ \ell = \frac{f}{2r_s} \]  \hspace{1cm} (2)

This is a (l+1)-d Mink (Radial spacetime).

To see this, consider \( ds_{\text{Min}}^2 = -\, dt^2 + d\xi^2 \)

Let \( \xi = \rho \sinh \psi \), \( T = \rho \cosh \psi \)  \hspace{1cm} (3)

\[ ds_{\text{Min}}^2 = -\, \rho^2 d\xi^2 + d\rho^2 \]  \hspace{1cm} (3) only over region I.

\[ \rho^2 = r^2 + T^2 \]

\[ \tanh \psi = \frac{T}{\ell} \]
Note: \( \mathcal{E} = T \) (\( \mathcal{E} > 0 \)) corresponds to \( \eta \rightarrow +\infty \) with \( \rho \rightarrow 0 \).

\( \mathcal{E} = -T \) (\( \mathcal{E} > 0 \)) corresponds to \( \eta \rightarrow -\infty \) with \( \rho \rightarrow 0 \).

Thus, the horizon of the black hole \( \rho = 0 \) is mapped to \( \mathcal{E} = \pm T \) (for \( \epsilon \rightarrow \pm \infty \)).

This mapping makes certain features manifest:

1) BH horizon is indeed non-singular.
2) We can extend the original metric to include all four patches of Minkowski space time as the near-horizon regions.

This mapping also tells us:

-
An observer at \( r > r_s \) (\( r_s \) is mapped to an observer following a world-line with \( \rho = \text{const} \) in Rindler spacetime, i.e. an observer in Minkowski spacetime following a \( \pm \) hyperbolic trajectory.

\[ \Sigma^2 + T^2 = \rho^2 = \text{const}. \]

It can be easily checked that such an observer is an accelerated observer with a constant proper acceleration

\[ \text{Accel} = \frac{1}{\rho}. \]

This matches well with the fact that a compact observer in BH \( \text{require} \) a force to hold it.

The E.T. surface is an event horizon for such an observer in mind as steady outside. I can teach him.
3) A free-fall (i.e. following radial geodesics) observer near the BH horizon \( \Rightarrow \) an inertial observer observes in Mink.

3) Region II should map to region inside \( r_\text{S} \) for Schwarzschild metric (near \( r_\text{S} \)).

It is clear that no information can propagate to an observer in region II. (future horizon)

4) The fact that Minkowski spacetime contains four regions indicates that we can also maximally extend the original Schwarzschild metric \( \mathcal{O} \) to include all four patches of the Minkowski spacetime as near-horizon region.

5) Clearly, observers in region II cannot influence events beyond \( x = -T \), (past horizon).
The fully extended black hole spacetime thus has the structure (r-t plane, each point corresponds to S^2)

Note:
1) $r \to \infty$ : $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ (Minkowski, but different from near-horizon one)

2) $\Pi$: can be considered as a mirror image of I. altogether two asymptotically flat regions.

3) $\Pi$: $r=0$ black hole singularity.

4) $IV$: $r=0$ "white hole" singularity (can receive signal)

5) $II$, $I\Pi$ nut present for BH spacetime results from collapse.
\[ ds^2 = f(-du dv) + r^2 dv^2 \]

\[ U = e^{-\frac{4\pi}{4\pi} \rho / r_s} \quad V = e^{\frac{4\pi}{4\pi} \rho / r_s} \]

\[ u = t - \rho r \quad v = t + \rho r \]

\[ dr_{\ast} = \frac{dr}{f} \quad e^{\frac{r}{r_s}} = e^{\frac{r}{r_s} f} \]

\[ du = \phi \frac{1}{2r_s} e^{-\frac{\varphi}{2r_s}} du \]

\[ dv = \frac{1}{2r_s} e^{\frac{\varphi}{2r_s}} dv \]

\[ ds^2 = -f 4r_s^2 e^{\phi / r} e^{-\varphi / r} du dv + r^2 dv^2 \]

\[ = -e^{-\frac{r}{r_s}} \frac{4r_s^3}{r} du dv + r^2 dv^2 \]

\[ U = T - \rho \]

\[ V = T + \rho \]
We thus find that:

\[ \xi^2 - T^2 = e^{\frac{r}{r_s}} \left( \frac{r}{r_s} - 1 \right) \]

\[ \frac{T}{\xi} = \tanh \frac{\xi}{2r_s} \]
Penrose diagrams:

Penrose diagrams provide an intuitively clear way to visualize the global geometry of a spacetime. Typically used for spherically symmetric cases, for which only r-t directions non-trivial.

Rules:
- Centrally map the whole spacetime to a finite portion of the r-t plane.

( The mapping preserves the directions of light rays and thus the causal structure of original spacetime.)

E.g. (1+1)-d Minkowski spacetime.

\[ ds^2 = -dt^2 + d\mathbf{s}^2 = -du \, dv \]

\[ u = T \pm \delta \]

\[ v = T \pm \delta \]
If one does this for the \( r-t \) plane of the extended Schwarzschild geometry:
1.2.3 Black hole temperature:

We will now show that the Schwarzschild BH has a temperature, as viewed by stationary observers outside BH.

\[ \text{CFT describing SM, } r = (\text{unit}) \]

There are many ways to show this. I will present the simplest, but also most sleek way.

Recall that in CFT, to describe a system at a finite $T$, we analytically continue to the time $t \rightarrow -i\tau$ to Euclidean signature and let $\tau \rightarrow 2\pi / T$ periodic:

\[ T = \frac{1}{\beta} \]

Conversely, if Euclidean description of a CFT is periodic in time direction, $\Rightarrow$ finite $T$. 
how analytically continue the BH metric to Euclidean

\[ ds^2_e = \frac{p^2}{4r_s} \, dt^2 + dp^2 + r_s^2 \, dv^2 \]

for \( r = r_s \)

\[ = \frac{p^2 \, d\Theta^2 + dp^2 + r_s^2 \, dv^2}{p^2 k^2 \, dt^2} \]

with \( \Theta = \frac{t - t_0}{2r_s} \)

at \( r = 0 \) (horizon)

The metric eq has a conical singularity unless

\[ \Theta = 0 \quad \Theta \text{ is periodic in } 2\pi, \quad \Theta = \Theta + 2\pi \]

in Lorentzian signature, horizon is regular,

\[ t_B \text{ with } p = \frac{2\pi}{4r_s} = 4r_s \frac{1}{t_B} \]

(From regularity at horizon)
Recall that $t$ is the proper time for an observer at $r = 0$

\[ T = \frac{\hbar k}{2\pi} = \frac{\hbar}{4\pi T_\text{BR}} = \frac{\hbar}{8\pi G m_\text{p} M} \sim \frac{m_\text{p}^2}{M} \]

For a stationary observer at $r = (\text{mt})$

\[ \frac{d\tau_{\text{loc}}}{d\tau} = f^\frac{1}{2}(r) \, d\tau \]

Euclidean:

\[ d\tau_{\text{loc}} = f^\frac{1}{2}(r) \, d\tau \]

\[ \Rightarrow \tau_{\text{loc}} \text{ is periodic with} \quad \beta_{\text{loc}} = f^\frac{1}{2}(r) \frac{2\pi}{k} \frac{1}{h} \]

\[ \Rightarrow T_{\text{loc}} = \frac{\hbar k}{2\pi} \frac{1}{f^\frac{1}{2}(r)} \quad \rightarrow 0 \quad \text{as} \quad r \rightarrow 0 \]
For an Rindler observer at $\rho = \text{const}$:

$$\dot{x}_\rho = \frac{p_k}{\rho}$$

$$\Rightarrow \frac{\rho \beta_{\text{loc}}}{\text{Unruh}} = p_k \cdot \frac{2\pi}{\hbar k} = \frac{2\pi \rho}{\hbar}$$

$$\Rightarrow T_{\text{loc}} = \frac{\hbar}{2\pi \rho} = \frac{\hbar A}{2\pi}$$

(Recall $A_{\text{prop}} = \frac{1}{\rho}$)

Thus in Minkowski space an accelerated observer will feel a temperature proportional to its acceleration.

The above derivation of $\beta T$, while simple, yet very powerful, we will use it again and again whenever talking about BHs. It is also too simple, as it hides various important physical points.

End at L3
Last lecture:

\[ ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2 \]

\[ f(r) = 0 \]

\[ \implies \tilde{\Omega}^2 + \beta \quad (\text{periodic } 0, \text{ angular variance}) \]

\[ \text{where} \quad \beta = \frac{\Omega r}{f r} = \frac{4 \pi^2}{f} \quad [\text{approximate}] \]

\[ \rightarrow \quad S^2 \]
1) The discussion of vacuum state in the BH geometry assumes a specific choice of vacuum state. 

AFT in a curved spacetime (like BH), there is no unique preferred choice of a vacuum state.

different choice \Rightarrow different physical properties.

In our derivation with Euclidean analytic continuation, 

\[ \text{BH + environment are in a thermal equilibrium,} \]

(e.g. not BH will at finite T radiates into vacuum)

The corresponding vacuum state describing this is called Hartle-Hawking state, which can also be specified by requiring AFT stress tensor (or wave functions) not singular at the horizon. (see linked definition)
2) Where does the finite temperature come from?

For simplicity, I will use the \textit{Rindler space} as an example. (generalization to full BH similar)

In this case, since Rindler is \textit{part of Minkowski} in different coordinates, a standard Minkowski vacuum is preferred and unique. (the only one in which \textit{plane wave} functions, stress tensors non-singular at horizon \( \xi = cT \))

Thus what derived earlier can be more precisely stated as:

The \textit{Rindler}\ in Minkowski spacetime appears to be a thermal state with temperature

\[
T = \frac{\hbar a}{2\pi} \quad (a = \frac{\rho}{p})
\]

or \( T = \frac{\hbar}{2\pi} \) in terms of

To a Rindler observer (with \( p = \text{const}\))
A digression:

- Harmonic oscillator at a finite $T$:

Consider a harmonic oscillator system given by a Hamiltonian $\hat{H} = \hbar \omega (\hat{a}^{\dagger} \hat{a})$ at finite $T$.

$$\langle \hat{X} \rangle_T = \frac{1}{Z} \text{Tr} (\hat{X} e^{-\beta \hat{H}}) = \frac{1}{Z} \int_{X(\omega) = X(\eta)} D\hat{X}(t) \hat{X} e^{-S[\hat{X}(t)]}$$

$$= \text{Tr} (\hat{X} \rho_\beta)$$

$$\rho_\beta = \frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle \langle n|$$

An alternative way (Umezawa, 1960's):

Consider two decoupled h.o. systems: $H_1 \otimes H_2$

$$H_1 \otimes H_2$$

Typical state $\sum_{m,n} a_{mn} |n\rangle \otimes |m\rangle_2$
2) \( |\Psi> \) is invariant under \( H_1 - H_2 \).

3) \( |\Psi> \) can be written as:

\[
|\Psi> = \frac{1}{\sqrt{1-e^{-\omega \rho}}} \exp (\frac{-\omega \rho}{2}) a^1 a^2 |0>, \Theta |0> \quad (\text{squeezed states})
\]

4) If we designate \( |\Psi> \) as some kind of 'vacuum' state, then one can show that

\[
b_1 = \cosh \Theta a_1 - \sinh \Theta a_2^+
\]

\[
b_2 = \cosh \Theta a_2 - \sinh \Theta a_1^+
\]

 annihilate \( |\Psi> \), i.e.

\[
b_1 |\Psi> = b_2 |\Psi> = 0
\]

\[
\cosh \Theta = \frac{1}{\sqrt{1-e^{-\omega \rho}}}
\]

\[
\sinh \Theta = \frac{e^{-\frac{1}{2} \beta \omega}}{\sqrt{1-e^{-\omega \rho}}}
\]
Consider
\[ \Psi = \frac{1}{\sqrt{2}} \left( |n\rangle_1 \otimes |n\rangle_2 \right) \]
(Entangled state)

Now suppose observables $\hat{X}(1)$ which only consists of operators in $\mathbb{C}^d$ act on $\Psi$.

\[ \langle \Psi | \hat{X}(1) | \Psi \rangle = \frac{1}{2} \sum_n e^{-\beta E_n} \langle n | \hat{X} | n \rangle \]

\[ = \text{Tr} \left( \hat{X} \rho \right) \]

By tracing over system 2, we obtain a thermal density matrix in 1.

\[ \text{Tr}_2 |\Psi\rangle\langle\Psi| = \frac{1}{2} \sum_n e^{-\beta E_n} |n\rangle_2 \langle n|_2 \]

Some additional remarks:

1) This formulation clearly applies to any quantum systems. (QFTs, ... )
Schrödinger representation of O(3):

Conventional Heisenberg Picture

$t=0$ \[ \Phi^0 \text{ configuration space} = \Phi(x) \]

$\psi$ wave functional \[ \Phi \left[ \Phi(x) \right] \in \mathcal{H} \]

Ground state \[ \langle \phi(x) | 0 \rangle = \Phi^0 \left[ \Phi(x) \right] \]

\[ e^{-i\hat{H}t} \phi(x) \]

\[ \langle \phi(x), t_2 | \phi(x), t_2 \rangle \]

\[ \phi_0 (x_2, x) = \phi_2 (x) \]

\[ \int D\Phi(x,x') e^{iS[\Phi]} \]

\[ \phi(x_1, x) = \phi_1 (x) \]
Now come back to Rindler space.

\[ ds^2 = -dT^2 + d\vec{x}^2 \]
\[ = -\rho^2 d\phi^2 + d\rho^2 \]
\[ \rho = \sqrt{T^2 + x^2}, \tan \theta = \frac{T}{x} \]

Consider a scalar field theory \( \phi \) in this spacetime.

\[ \mathcal{H}_{\text{Rindler}} = \{ \Phi[\phi_\rho(x)] \mid \phi_\rho : \mathbb{R} \rightarrow \mathbb{C}, \theta = \frac{\phi_\rho}{\rho}, \rho > 0, T = 0 \} \]

\[ \mathcal{H}_{\text{Minkowski}} = \{ \Phi[\psi(x)] \mid \Phi[\psi(x)] = (\Phi_c(x)\Phi_\rho(x)) \text{ for all } x \in \mathbb{R} \} \]

\[ = \mathcal{H}_{\text{Rindler}} \otimes \mathcal{H}_{\text{Rindler}} \quad \mathcal{H}_M : \text{w.r.t.} \quad T \]

Now consider the vacuum state w.r.t. to \( \mathcal{H}_M \) in \( \mathcal{H}_{\text{Minkowski}} \), which can be obtained from path integral using standard procedure.
Digression: Vacuum state from Euclidean path integral

Consider a QM system with $Q(x)$ and Hamiltonian $H$:

\[
\langle Q, 0 | 0, -iTE \rangle = \langle Q | e^{+iHT} | 0 \rangle = \sum_n \psi_n(Q) \psi_n^*(0) e^{+iE_n T}
\]

Now take $T \to -iE$ and $E \to 0 - i$

\[
\lim_{T \to -iE} \langle Q, 0 | 0, -iTE \rangle \quad \approx \quad \psi_0(Q) \psi_0^*(0) \quad (E = 0)
\]

\[
\Rightarrow \quad \psi_0(Q) = \int_{TE \in (-iV, 0)} Q(TE) \ e^{-SE [Q(TE)]}
\]

Similar with a QFT

\[
Q(TE) \rightarrow \Phi(TE, \hat{x})
\]

\[
\Phi(TE = 0, \hat{x}) = \Phi(x)
\]

\[
\int d\Phi(TE, \hat{x}) \ e^{-SE[\Phi]}
\]

\[
\Phi(TE < 0) = \Phi[\Phi(x)]
\]
90° to Euclidean signature.

\[ T \rightarrow -c T_E \]
\[ \Pi \rightarrow -c \theta \]

\[ ds^2 = dT_E^2 + d\pi^2 = r^2 d\Omega^2 + dr^2 \]

\[ r = \sqrt{T_E^2 + \pi^2} \quad \text{and} \quad \tan \theta = \frac{T_E}{\pi} \]

\[ \theta \in (0, 2\pi) \]

Euclidean analytic continuation of Minkowski and Rindler completely coincide. They correspond to different formulations of \( R^2_E \).

\[ T_E = 0 \quad \text{surfaces for Euclidean Minkowski} \]

In particular, Euclidean analytic continuation of Kleinian

both \( R \) (for \( \theta = 0 \)) and \( L \) (for \( \theta = \pi \)).

of Euclidean continuation of Rindler.

The vacuum state wave function for \( Hu \) can be written as...
Now we define the vacuum state of Minkowski vacuum:

\[ \Psi_0 [\phi(x)] = \Psi_0 [\phi_L(x), \phi_R(x)] \]

\[ = \int D\phi [T_0, x] e^{-S_{E} [\phi]} \]

\[ \phi_L(\vec{r}) \leftarrow r \rightarrow \phi_R(\vec{r}) \]

\[ = \int D\phi (\theta, \rho) e^{-S_{E} [\phi]} \]

\[ \phi (\theta = 0, \rho) = \phi_R \]

\[ \phi (\theta = \pi, \rho) = \phi_L \]

\[ \Rightarrow \Psi_0 [\phi_L, \phi_R] = \langle \phi_R | e^{-\pi HR} | \phi_L \rangle \]

\[ = \sum_n e^{-\pi E_n} \chi_n [\phi_R] \chi_n [\phi_L] \]  \hspace{1cm} (\ast) \]

\( |n\rangle \) a complete set of eigenstates of \( HR \)

\[ \chi_n [\phi] = \langle \phi | n \rangle , \text{ the corresponding wavefunction} \]
\[ \Psi_0[\phi_L, \phi_R] = \sum_n \frac{1}{\sqrt{n!}} e^{-\frac{\pi}{2} E_n} \chi_n[\phi_R] \chi_n^*[\phi_L] \otimes \chi_n^*[\phi_L] \otimes \chi_n[\phi_R] \]

with \( \hat{\chi}_n[\phi_L] = \chi_n^*[\phi_L] \)

i.e. \( \hat{\chi}_n \in \mathfrak{H}_{\text{Kink}} \), whose time evolution is generated is opposite to the Kink space we stated with.

\[ \langle 0 | \hat{O} | 0 \rangle_{\text{min}} = \frac{1}{\alpha} e^{-\pi E_n} \left( \begin{array}{c}
\ln \gamma_{\text{Kink R}} \\
\ln \gamma_{\text{Kink L}}
\end{array} \right) \]

(\( \mathfrak{H} \))

This is precisely the entangled state we discussed earlier which gives an equivalent description of thermal state. Or in \( \mathcal{R} \) region: i.e. for any observable \( \hat{O} \)\

\[ \langle 0 | \hat{O} | 0 \rangle_{\text{min}} = \text{Tr}_{\mathfrak{H}_{\text{Kink L}}} \left( e^{-2\pi \Theta \hat{O}} | 0 \rangle \langle 0 | \right) \quad \text{with} \quad \beta = 2T \]
The previously somewhat artificial construction of $\Phi_0$ now has a beautiful geometric interpretation:

Thus the thermal nature of a Rindler space observer in the Mink vacuum can be attributed to that it has no access to the $\omega L$ region. When $\delta a^+_j$ out all the d.o.f. in the L region

$\Rightarrow$ thermal state in $R$-region.

If we expand:

$$\phi_R = \sum_j \alpha_j (a_j u_j + a_j^+ u_j^*)$$

$u_j$: a basis of solutions for $\phi_R$, with frequency $\omega_j$

Then

$$10 \gamma_{\text{min}} = \frac{1}{\sqrt{2\pi} \hbar} \prod_j \exp \left( e^{-i \pi \omega_j} a_j a_j^+ \right) \left| \Omega \right|_{L(O)}$$
Similarly, the annihilation operators of the Minkowski vacuum are related to those of Rüdels vacuum by a Bogoliubov transformation.

The above story for Rüdels space can now be immediately generalized to Schwarzschild metric:

\[ |0\rangle_{\text{HH}} = \text{path integral over half of Euclidean analytic continuation of BH space-time.} \]

Euclidean \( \times S^2 \)
1.2.9 Black hole entropy.

We have found a Schwarzschild BH with mass $M$, has temperature: $T = \frac{\hbar k}{2\pi l} = \frac{\hbar}{8\pi G_N M}$.

Now recall thermodynamic relations: $T = \frac{\hbar}{4\pi G_N l}$

We can integrate it to get: $T(E) = M$

$$S(E) = \int \frac{dE}{T(E)} = \int \left(\frac{\hbar}{8\pi G_N l E}\right)^{-1} dE$$

$$= \frac{1}{\hbar} 4\pi G_N E^2 + \text{const} \ (\text{taken to be 0})$$

$$n = \frac{4\pi R^2}{4\pi G_N} = \frac{A_{m}}{4\pi G_N}$$
\[ S_{\text{BH}} = \frac{A_{\text{H}}}{4\hbar G_N} \]

\[ T_{\text{BH}} = \frac{\hbar c}{2l} \]

Add specific heat

This result is not a coincidence and in fact applies to all BHs in Einstein gravity + matter fields.

1.2.5 General BHs. (most quote results)

No hair theorem: A stationary, asymptotically flat black hole is characterized by its

1) mass \( M \)

2) angular momentum \( (\text{Kerr}) \, J \)

3) electric + magnetic charge \( (\text{Q}, \text{Q}) \)

Consider a complex matter system, say a star which collapses to form a BH. Then no-hair theorem
Note: \textit{Specific Heat.}

\[ T = \frac{\hbar}{8\pi G N M} \]

\[ M \uparrow, \quad T \downarrow \]

i.e. the system has negative specific heat.

\[ \Theta C = T \Theta \left( \frac{\delta f}{\delta T} \right) = \frac{\delta F}{\delta T} \]

\[ \Theta = -\frac{\hbar}{8\pi G N T^2} = -\frac{8\pi G N M^2}{\hbar} \]

\[ C \rightarrow 0, \quad T \rightarrow \infty \]
sends the final state is unique (given by conserved charges of long range interactions).

- Four laws of DA mechanics

1st law: \[ \text{d}M = \frac{k}{8\pi G_N} \text{d}A + \Omega \text{d}J + \mathcal{E} \text{d}Q \]

\( \Omega \): angular frequency at the horizon

\( \mathcal{E} \): electric potential --- (assume that at \( r = \text{as} \))

2nd law: horizon area never decreases classically

(positive energy condition \( \Theta + \) no naked singularity)

3rd law: surface gravity of a BH cannot be reduced to zero in a finite \( \text{# of steps} \).
These laws became the standard laws of thermodynamics if we identify:

\[ T = \frac{\hbar k}{2\pi}, \quad S = \frac{A}{4\pi Gm^4} \]

In particular:
\[ dM = T dS + \partial dJ + \mathcal{E} dQ. \]

Bekenstein (1972–1974):
\[ S_{\text{BH}} \propto A \]

Motivation:
- 2nd law of thermodynamics violated in gravitational collapse.
- Entropy conserved at the event horizon (or just throw things into a BH)
- Generalized 2nd law:
\[ S_{\text{Total}} = S_{\text{BH}} + S_{\text{matter}} \]
\[ dS_{\text{Total}} \geq 0 \]

(because Hawking radiation, not taken seriously)

If we accept a BH as a thermal object, then GSL is automatic.

Classical limit:
\[ \hbar \to 0 \quad \text{GSL nexus} \quad T \to 0 \quad \text{and law} \quad S \to 0 \quad \text{and law} \]
In string theory, many different types of black holes, black branes have been found. These formulas work all of them.

(again Einstein gravity + matter)

Also works for other asymptotics.

To string theory, there is higher derivative corrections to Einstein gravity (there are all many other generalizations).

Including them, entropy is a longer given by

area, but can still by evaluating some geometric quantities at the horizon.
Some puzzles:

BH:
Classically, $T=0$, absorbs everything, no hair theorem.

QM:
Finite $T$, nonzero entropy.

(a) Does BH entropy have a statistical interpretation?
(b) Should BH be treated as a system consistent with our usual laws of quantum mechanics and perturbation theory?

(a) Yes. This implies that a BH hole should have allowed internal states of order $\frac{A}{\hbar G M}$

$N \propto E$

For a BH like the Sun, $r_s \sim 1.5 \text{ km}$

$N \propto \exp[1077]$
(b) Hawking's information loss paradox

Tough version: imagine throwing a book into a BH, then BH will send out thermal radiation. Information is lost.

More precisely, consider a state which is a pure state collapsing to form a BH, which radiates thermally.

Hawking's argument goes as far as $M > 8\hbar \nu$ (thermal radiation) with small corrections, but once $M$ becomes $O(\hbar \nu)$, it will be too late for all the information to go out.

Hawking: BH then sends QM.

QM has to be modified in the presence of a BH.
1.3 Holographic principle

Important implications if we treat BH as an ordinary object. Any object whose dive is large

A sufficiently massive star will eventually collapse to form a BH.

Any system with a mass \( m \) which is confined inside a region \( R < \frac{2GM}{c^2} \) will necessarily become a BH.

We will now argue this universal fact (BH entropy)

\[ \Rightarrow \] holographic principle. (Heuristic)

Consider an isolated system of mass \( E \) and entropy small in an asymptotic flat spacetime.

Introduce \( A \) to be the area of the smallest sphere that fits around the system.
Define $M_A$ to be the mass of a Black Hole of the same surface area.

Clearly: $E < M_A$ as otherwise the system would be already a BH.

Now let us collapse a shell of energy $\Delta M \rightarrow M_A - E$ onto the system $\Rightarrow$ BH of mass $M_A$

$$S_{BH} > S_{\text{matter}} + S_{\text{shell}}$$

$$\Rightarrow \quad S_{\text{matter}} \leq S_{BH} = \frac{A}{4\pi \hbar c^2}$$

Maximal entropy inside a region bounded by area $A$ is

$$S_{\text{max}} = \frac{A}{4\pi \hbar c^2}$$
New recall the meaning of entropy in quantum statistical physics:

\[ S = - \text{Tr} \rho \log \rho \]

\( \rho \): density matrix

Consider a system with a \( N \)-dimensional Hilbert space.

E.g., if we have a system of \( n \) spins,

\[ N = 2^n \]

With the dimension of \( \mathcal{H} \) for a single h.o. is infinite.

For any quantum system, \( \mathcal{H} \) below some energy scale is finite volume.

(\( \text{finite volume} \)) always finite-dimensional.

Define:

\[ \# \text{ of } \text{d.o.f.} \propto \log N \]
Now one can show that the maximal entropy of a system with $N$ states is

\[ S_{\text{max}} = \log N \]

\[ \text{i.e. } \# \text{ of d.o.f. } \leq S_{\text{max}} \]

We thus conclude that the \# of d.o.f. inside a region of area $A$ is bounded by

\[ \# \text{ of d.o.f.} \leq \frac{A}{4\pi G_{\text{N}}} = \frac{A}{4\pi \rho^2} \]

\[ \text{Holographic Principle} \]
This is a very surprising statement, clearly violated in a non-gravitational theory.

Example: Consider we are dealing with a lattice of spins, with lattice spacing $a \geq \epsilon_p$

Then
\[
\text{total # of spins} = \frac{V}{a^3} = \frac{\frac{4}{3} \pi A}{a^2} \frac{L}{a} \\
\gg \frac{\pi^2}{6} \frac{A}{\epsilon_p^2}
\]

\[
N_{\text{state}} = 2^V \frac{V}{a^3}
\]

\[
S = \frac{V}{a^3} \log 2 \geq S_{\text{HH}}
\]

2) QFT: Clearly have even more degrees of freedom, h.o.
A BH saturates the bound.

An implicit assumption: BH entropy can be understood statistically as in any quantum system.

Thus, A.G. leads to a huge reduction of # of d.o.f.

Holographic principle: A region of boundary area A can be fully described by no more than $\frac{A}{4\hbar G_n}$ degrees of freedom. 

It should be possible to at least describe the system its boundary with (bit of d.o.f. per Planck area).

BH has brought A.G. to a macroscopic level.
1.4 Large $N$ expansion of gauge theories

We will now look at clues to gauge/gravity duality from field theory side.

Consider QCD: $SU(3)$ gauge theory + fundamental quarks

$$ L = \frac{1}{g^2} \int \left[ -\frac{1}{4} \text{Tr} \, F_{\mu \nu} F^{\mu \nu} - i \bar{\psi} (\gamma^\mu D_\mu - m) \psi \right] $$

$$ \epsilon_{SU(3)} $$

$$ Q_\mu : \partial_\mu + i A_\mu, \quad A_\mu = A_\mu^a T^a, \quad 3 \times 3 $$

$$ \psi : \text{Weyl 3-component vector} $$

$$ (\text{color index}) $$

The theory is hard to solve; coupling becomes strong in $1/R$.

There is no coupling constant, only a scale $\Lambda_{\text{QCD}} \approx 250 \text{ MeV}$

(dimensional transmutation)

No small parameter to expand.

$\Box$ Still an open problem.
1974, 't Hooft: take the # of color: \( N = 3 \) as a parameter.

Consider \( N \rightarrow \infty \) limit, \( \frac{1}{N} \): small parameter.

do \( \frac{1}{N} \) expansion.

A generic idea: unfortunately, QCD cannot be solved this way.

but other nice things appear in this approach.

but we will see: \( \frac{1}{N} \) expansion \( \rightarrow \) string theory.

The structure which will arise in the \( \frac{1}{N} \) expansion has nothing to do with gauge, \( \Phi \), field theory, only has to do with the matrix nature of \( \Phi \).

We will thus simplify our discussion by forgetting the Lorentz structure of \( A \) & also for the moment concentrate on \( \Phi \) as adjoint fields only.
we will consider:

$$\mathcal{L} = -\frac{1}{\Theta^2} \bar{\Theta} \text{Tr} \left[ + \frac{1}{2} \Theta (\partial A)^2 + \frac{1}{4} A^4 \right] \quad (2)$$

A: N x N hermitian matrix

Feynman rule:

$$\langle A_{a_b}(x) \ A_{c_d}^c(y) \rangle = g^2 \delta_{a_d} \ \delta_{b_c} \ \delta(x-y)$$

\[ \begin{array}{c}
\alpha \\
\alpha
\end{array} \quad \text{to make the index contraction manifest}
\]

\[ \begin{array}{c}
\alpha \\
\alpha
\end{array} \quad \text{direction of arrow: upper index to lower index}
\]
Let us now work out \( N \) - counting for a generic Feynman diagram. They arise from undacity propagators.

Consider first vacuum diagrams (no external legs): refer to evaluate partition function, start with lowest order ones in perturbation theory.

![Diagram](image)

Each contracted index line gives a factor of \( N \).

Altogether \( N^3 \) = only one contracted line \( \Rightarrow N \)

(a) Can be drawn on a plane without crossing lines.

(b) Cannot be drawn
(a) is called a planar diagram.

(b) non-planar diagram

Note that each edge can be drawn on a torus without crossing lines.

For both (a) and (b): power of $N = \#$ of faces in each diagram after we degenerate straighten it out by putting it on an appropriate surface.
A slightly more complicated example:

\[ N^4 \text{ as can be easily checked using double line notation.} \]

\[ N^2 \]

( planar )

( non-planar )

( on torus )
Fact: Any orientable two-dimensional surface is classified topologically by an integer, called genus.

The genus is equal to the number of 'holes' the surface has.

- (or plane) genus 0
- torus genus 1
- genus 2

The genus is also given by the maximal number of cuttings along non-intersecting closed simple curves (non-self-intersecting) without rendering the surface disconnected.

Euler number: \( \chi = 2 - 2g \) (topological invariant)
Claim 1: For any non-planar diagram, there exists an integer $h$, so that the diagram can be straightened out (i.e. made non-crossing) on a genus-$h$ surface, but not on a surface with a smaller genus.

Claim 2: For any non-planar diagram, the power of $N$ coming from contracting propagators is given by the <sup>number of faces on such a genus-$h$ surface</sup>. (i.e. number of disconnected regions separated by Feynman diagram.)

Both claims are self-evident after a bit of practice. (Landau: Life is too short for proofs!)
A Feynman diagram has the following dependence on \( q^2 \) and \( N \):

\[
A \sim (q^2)^E (q^2)^{-V} N^F
\]

- \( E \): \# of propagators
- \( V \): \# of vertices
- \( F \): \# of faces

This, however, does not give a sensible \( \frac{1}{N} \) expansion.

Need to find diagrams of largest powers of \( N \).

But it does not exist. Can find diagrams with any value of \( F \), no matter how large.

Hooft: \( N \to \infty \), \( \Theta \lambda \equiv q^2 N = \text{fixed} \).

\[
A \sim (q^2 N)^{E-V} \Theta N^{F+V-E} = \lambda^{E-V} N^F
\]
\[ L : \text{\# of loops} \]

\[ \chi = F + V - E \]

*Euler number for the genus-0 surface.*

Each Feynman diagram can be considered as a partition of the surface, i.e., separate it into polygons.

**Theorem.** Given a surface composed of polygons with \( F \) faces, \( E \) edges, and \( V \) vertices

\[ \chi = 2 - 2h \]

\( \Rightarrow \) \( \chi = \lambda \cdot N^{2-2h} \)
Thus to leading order in large $N$, we should take

$h=0$: i.e. planar diagrams, the sum of which gives

\[ N^2 \left[ C_0 + C_1 \lambda + C_2 \lambda^2 + \ldots \right] = N^2 f_0(\lambda) \]

In general, include higher order $\frac{1}{N^2}$ corrections.

\[ \log Z = \sum_{n=0}^{\infty} N^{2-2n} f_n(\lambda) \quad (\text{all vacuum diagrams}) = N^2 f_0(\lambda) + f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \ldots \]

↑ 🔄 planar  ↑ 🔄 genus-1  ↑ 🔄 genus-2.

Thus, in the 't Hooft limit, $\frac{1}{N^2}$ expansion

\[ \iff \text{topological expansion} \]

(expansion in terms of topology of diagrams)
There is a very simple way to understand that at leading order $\log z = O(N^2)$

$$z = \int DA \ e^{s L}$$

$$L = \frac{1}{g^2 N} \ Tr \left( \frac{1}{2} A^2 + \frac{1}{4} A^4 \right) \sim O(N^2)$$

This observation + general nature of arguments leading to $\Rightarrow$ any adjoint fields

3. Works for any Lagrangian of the form

$$L = \frac{\lambda N}{g^2 N} \ Tr (\cdot \cdot \cdot)$$

$\uparrow$ matrix-valued fields, any polynomial interactions

Including YM theory also works at finite temperature...
For any theory with adjoint fields of the form:

\[ L = \frac{N}{\lambda} \text{Tr} \left( \cdots \right) \text{ sum of field operators} \]

\[ \lambda = 't \text{Hooft coupling} \]

\[ \log z = \frac{1}{2} n_{>0} N^{2-2h} f_n(U) \]

\[ = N^2 f_0(U) + f(U) + \frac{1}{N^2} f_2(U) + \cdots \]

\[ \uparrow \]

sum of all planar diagrams.

not known how to do it for $\mathcal{N}=1$.

possible for a matrix integral or AM.
If there are other matter fields in adjoint rep.

\[ O_n = \text{Tr} \left( F^n \right) \]

Single trace operators

\[ T^\alpha (F) \]

Multitrace ones

Here let me consider local operators (can be local or \( n \)-point functions)

\[ \text{General observables} \]

After looking at general observables, having in mind YM theory

\[ \text{Correlation functions} \]

\[ \text{Large } N \text{ expansion of theory} \]
Since multi-trace operators can be obtained from products of single-trace ones (at the same spacetime points), it is natural to look at their correlations can be obtained from those of single-trace ones.

\[
\langle 0 | O_1 (x_1) O_2 (x_2) \cdots O_n (x_n) | 0 \rangle_c
\]

For this purpose, we look at the generating functional.

\[
Z [J_1, \ldots J_n] = \int \mathcal{D} A_k \cdots \exp \left[ -S_0 - N \sum_i \int \mathcal{D}^{(0)} O_i (x) \right]
\]

\[
\langle 0 | \cdots | 0 \rangle_c = \left. \frac{\delta \log Z}{\delta J_1 (x) \cdots \delta J_n (x)} \right|_{N^a}
\]
Self has the same form as we discussed earlier

\[
N \text{ Tr} \left( \ldots \right) \quad \{ \text{sum of local operators} \}
\]

\[
= \log Z \left[ \{ J_i \} \right] = \sum_{h=0}^6 N^{2-2h} f_h (\lambda, \ldots)
\]

\[
\Rightarrow \quad \langle 0_1 0_2 \ldots 0_n \rangle_c \sim N^{2-n} (1 + O(\frac{1}{N^2}))
\]

e.g.

\[
\langle 1 \rangle \sim O(N^2)
\]

\[
\langle 0 \rangle \sim O(N)
\]

\[
\langle 0_1 0_2 \rangle_c \sim O(N^0)
\]

\[
\langle 0_1 0_2 0_3 \rangle_c \sim O(N^{-1}) \quad \ldots
\]

Again, leading order contribution from planar diagrams.
physical implications.

(a) In YM theory, a gauge invariant operators formed from gauge fields, like $O = Tr F^a$ (schematic form).

Create "glue balls" when acting on the vacuum.

$$0 | 0 > \leftrightarrow \text{glue balls}$$

If we interpret $O_i(x) | 0 >$ as creating

\[ \uparrow \]

single-trace

single-particle states, then multiple-trace operators

like

$$0_1 O_2(x) | 0 > \quad 2- \text{particle}$$

or

$$0_1 O_2 O_3(x) | 0 > \quad 3- \text{particle}$$

Can be considered as creating multi-particle states.

This outline can be made precise in the large-$N$

limit.
(i)  \[ \langle 0; 0_j \rangle \propto O(N^0) \]

we can diagonalize them so that \[ \langle 0; 0_j \rangle \propto \delta_{ij} \]

(ii)  \[ \langle 0; (x) \; 0^2(y) \rangle_{\alpha} \propto O(N^{-1}) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \]

i.e. no overlap between single and multi-particle states.

(iii)  \[ \langle 0; 0_1(x) \; 0_2(y) \rangle \]

\[ = \langle 0; (x) \; 0_1(y) \rangle \langle 0; 0_2(y) \rangle \]

\[ \propto O(N^0) \]

\[ + \langle 0; 0_1(x) \; 0_2(y) \rangle \]

\[ \propto O(N^{-2}) \]

From this argument, we see that this notion of single and multi-particle glue ball states become ambiguous at finite \( N \).
(b) With this interpretation, then

\[ \langle 0_1, 0_2, \ldots, 0_n \rangle \sim N^n \sim N^{2-n} \]

we will now argue that scattering amplitudes resemble like tree amplitudes involve only tree-level interactions among "glue ball" states.

(i) There are no more than one-particle intermediate states. If there are loops, e.g.,

\[ \sim \text{two-particle states.} \]
e.g. consider $<0, O_2 O_3 > \propto O(N^{-1})$

insert complete set of states at all possible planes.

Due to large $N$ counting, all states other than single-particle ones are suppressed by $N^{-1}$.

e.g. $<0, O_4 O_5 > < O_6 O_7 O_8 O_9 > \propto N^{-3}$

compared to $<0, O_i > < O_i O_k O_l > \propto N^{-1}$

(iii) $N$-dependence for $n$-particle amplitudes is consistent with tree-level amplitudes for a theory of glueballs with coupling $g_s \propto \frac{1}{N}$

\[
\begin{align*}
\text{or higher allowed} & \quad \propto g_s^{-2} \\
\text{or higher allowed} & \quad \propto g_s^{-2}
\end{align*}
\]
(c) Fluctuations of glueballs suppressed.

Suppose \( \langle 0 \rangle = 0 \sim O(N) \)

then consider the variance of \( \langle 0 \rangle \)

\[
\langle 0^2 \rangle - \langle 0 \rangle^2 = \langle 0^2 \rangle_c \sim O(1)
\]

\[
\frac{\langle 0^2 \rangle_c}{\langle 0^2 \rangle} = \frac{\langle 0^2 \rangle_c}{\langle 0 \rangle^2} \sim \frac{1}{N^2} \rightarrow 0
\]

Also easily

\[
\langle 0_1 \ 0_2 \ 0_3 \ \rangle \sim \langle 0_1 \rangle \langle 0_2 \rangle \langle 0_3 \rangle \sim O(1)
\]

\[
\langle 0_1 \ 0_2 \ 0_3 \ \rangle_c \sim O(N^2)
\]

\[
\text{d^2-connected part dominates } \sim O(1)
\]

like in a classical theory.
Summary: \((a) + (b) + (c) \Rightarrow\)

In the leading order in \(1/N\) expansion, we obtain a classical theory of glue balls, with interactions among glue balls given by \(g_s \propto 1/N\).

\[\text{gauge theory} \quad N \to \infty \quad \text{finite} \quad = \quad \text{glue ball theory} \quad \frac{1}{N} \to \infty \text{ as } N \to \infty\]

Large \(N\) expansion as a string theory

String theory: describing propagation of strings.

Qm: \[\int e^{-S} \quad \text{vacuum amplitude for all particle trajectory.}\]
Vacuum amplitude for strings propagating in certain spacetime:

$$\log Z = \sum \quad e^{-S_{\text{string}}}$$

(S_{\text{string}} = \text{area of surface in some target space})

$$= \sum \quad g_s^{2h-2} \quad \sum \quad e^{-S_{\text{string}}}$$

where $g_s$ is string tension, $F_h$ is the field strength, and $R$ is the curvature radius of target space.
\[ \frac{1}{g_s} + O(g_s^0) + O(g_s^2) + \ldots \]

[free-level]

[1-loop]

classical string

We see that:

\[ \log z = \sum_{k=0}^{\infty} N^{2-2h} f_k(u) \iff \log z = \sum_{k=0}^{\infty} g_s^{2h-2} F_k \]

[large N gauge theory]

\[ \frac{1}{N} \iff g_s \]

\[ \langle 01 \ 02 \ 03 \rangle \sim O(\frac{1}{N}) \iff \]

\[ \langle 01 \ldots \ 0n \rangle \sim O(N^{2-n}) \iff \text{free amplitudes with} \]

n external strings

end of L.7
Last lecture:

1) gauge theory in the \( N \to \infty \) limit

\[ \leftrightarrow \]

classical theory of glue balls

2) \( 1/N \) expansion of gauge theory

\[ \leftrightarrow \]

perturbative expansion in string theory

\[ \frac{1}{N} \] vs. \( g_s \)

\[ \left( \right) \]

expansion in terms of topology of 2d closed surfaces

I will now elaborate this relation a bit more.
More about string theory:

Consider propagation of a string in a 10-dimensional space-time:

Classical action:

\[ S_{\text{prop}} = \text{constant} \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^n}{d\tau}} \, d\tau \]

(E.g., \( g_{10} = g_{11} \))

[Euclidean]

\[ ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \]

[Euclidean]
For a string

\[ S_{\text{Net}} = \frac{1}{2 \pi a} \int \frac{dA}{z} \]

\[ T = \frac{1}{2 \pi a} = \text{string tension} \quad ( \text{mass per unit length} ) \]

\[ dA = d\alpha d\tau \sqrt{\varepsilon + \dot{\varepsilon} d \varepsilon} \]

\[ \varepsilon = g_{\mu\nu} \partial \alpha^{\mu} \partial \alpha^{\nu} \quad (a, b = 0, \infty) \]

(induced metric on worldsheet)

Note: \( T \) does not depend on choice of \( \frac{\varepsilon}{d\varepsilon} \).

String theory: All of \( T \) excitations of string and their scattering etc.

\[ \Rightarrow \text{Graviton, QG} \]
To quantize \( \mathcal{A} \), one can use path integral:

\[
\mathcal{Z}_{\mathcal{A}} = \int \mathcal{D}\mathbf{x}(\sigma,t) \ e^{-i \mathcal{S}_{\mathcal{W}} \left[ \mathbf{x}(\sigma,t) \right]}
\]

(Notice: convenient by going to Euclidean signature)

\[
= \sum_{\text{sum over all surfaces \( \Sigma \)}} e^{-\mathcal{S}_{\mathcal{W}}} \quad \text{(for vacuum processes)}
\]

\[
= \sum_{\text{sum over topologies}} \sum_{\text{genus } h} \sum_{\text{topology \( \sigma \) for a given genus } h} e^{-\mathcal{S}_{\mathcal{W}}}
\]

\[
= g_{A}^{-2} + g_{g_{3}}^{10} + g_{b} + \ldots
\]
Remarkable facts:

Summing over all interfaces includes interactions of strings. This fully specifies the nature of interactions. There is no freedom of making arbitrary (non)choices.

Given non-polynomial action of \( \mathcal{O} \), \( \mathcal{O} \) is not convenient for explicit calculations.

One can rewrite \( \mathcal{O} \) as (polyakov action)

\[
S_{\text{P\left[ \mathcal{E}, \mathcal{Y} \right]}} = \int \frac{d^2 \sigma}{4 \pi} \sqrt{g} \mathcal{Y}^{ab} \, d\mathcal{X}^a \wedge d\mathcal{X}^b \quad (8) \]

with additional world sheet metric

\[
\text{a set of an independent finds. (without derivatives)}
\]

\[
\text{symmetric} \quad a, b \quad \text{intrinsic WS metric}
\]
Elementarily, by using its own e.o.m. \( S_{NG} \)

Thus, one can write the path integral as

\[
Z_{NG} = \int \mathcal{D} \varphi_a (0, 2) \mathcal{D} \Delta_{\varphi} (0, 2) e^{-S_p (\varphi, \Delta)}
\]

\( \sum_0 \)

\( \xi \) may be different from \( \varphi \), but use \( \xi \) as the definition of \( \varphi \).

\( \xi \) has a number of gauge symmetries:

(a) 2-d coordinate transformations (2d diffusion)

\[
\Delta_{\varphi} (t', 0') = \Delta_{\varphi} (t, 0)
\]

\[
\frac{\delta \varphi_a}{\delta c^b} \frac{\delta \varphi_b}{\delta c^a} \delta_{\varphi} (0', 2') = \delta_{\varphi} (0, 2) \quad \delta^a (0, 2)
\]

for any \( 0 \leq \xi \leq 0', 2', 2 \)
(b) Weyl transformations.

\[ \Delta^W \rightarrow \Delta^W \]

\[ Y_{ab}(\sigma; \tau) \rightarrow e^{2\omega(\tau, \sigma)} Y_{ab}(\tau, \sigma) \]

for any function \( \omega(\tau, \sigma) \).

We can also use (a) + (b) as symmetry principle which \( \alpha \) (uniquely) determines \( \Theta \).

There is in fact one additional term which is consistent with (a) + (b):

\[ \text{Scalar} = \frac{\Lambda}{8\pi} \int d\sigma d\tau \, F_{ab} R \]

\( R \) is Ricci scalar constructed out of \( Y_{ab} \).

\[ \lambda = \text{arbitrary constant} \]

(\( \Theta \) is clearly invariant under (a) under (a) up to an arbitrary derivative.)
Thus altogether \( D = \text{H \times Weyl} \Rightarrow \)

\[
S = \int d\sigma \sqrt{g} \left[ \frac{1}{4\pi} \Gamma^{\alpha \beta \gamma} \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu} g_{\mu \nu} + \frac{\lambda}{4\pi} R \right]
\]

\[ Z_m = \int D\varphi \ D\bar{\varphi} \ e^{-S[\varphi, \bar{\varphi}]} \]

Note: in \( d=2 \) the Einstein-Hilbert term \( \mathcal{O} \) is in fact topological (locally total derivative):

\[
S_{\text{Euler}} = \lambda \chi
\]

\( \chi = 2 - 2g \) : Euler \# only depends on topology of the surface.
\[ Z_\text{total} = \sum_{n=0}^{\infty} e^{\lambda (2h-2)} Z_n (d') \]

Identity : \[ q_s \propto e^{\lambda} \]

\[ Z_n (d') = \int D\gamma D\pi e^{-S_{\pi} [\gamma, \pi]} \]

\( \text{genus} \ h \)

\( \text{modular} \)

---

**Note:**

(a1 + b1), altogether 3 gauge d.o.f.

\( \gamma \) a (c2, 2), 3 independent components

\[ = \quad \text{one can gauge fix } \gamma \text{ completely, no dynamical d.o.f. associated with } \gamma \]

(Caveats)

\( \Rightarrow \) after gauge-fixing, one obtains a 2d \( U(1) \) for even (c1) (+ ghosts)
Physical spectrum (include graviton) 
Scattering amplitudes etc.

Supersymmetry: add some 2d terms to $Sp$.

(Internal excitations of WS)

It turns out:

1) A consistent gauge fix is for Weyl

Symmetry is only possible for $\mathbb{D} = 26$ (bosonic)

and $\mathbb{D} = 10$ (for supersymmetry).

2) For $\mathbb{D} = 26$, or 10, $\Box_0 \delta_{ab} = e^{2\Phi} \delta_{ab}$

\( \Phi \): does not decouple (Weyl anomaly)

It is believed that treating $\Phi$ as a dynamical field

(2D-while fixed)
on WS can again save the day (only know how to

do it in $d=0$

Explicitly, $d=1$)

\( \Sigma^a + \Phi \)

(opens up higher dim.)
Now let us go back to the question:

\[ \frac{1}{N} \text{ expansion of gauge theory} \quad \leftrightarrow \quad \text{perturbative expansion of a string theory} \]

\[ \sum_{n=0}^{\infty} \frac{1}{N} N^{2-2n} f_n(n) \]

\[ \sum_{n=0}^{\infty} e^{1/2} \left( \frac{2^1}{e^2} \right) \]

(\( R_c \): curvature radius)

\[ f_n(n) \quad \leftrightarrow \quad g_5 \]

\[ f_n(2^1) \]

\[ F_n = \int \text{d}x \text{d}x \text{e}^{-S_p} \]

\[ \text{gauge-h surfaces} \]

\[ \int \text{d}x \text{d}x \int \text{d}x \text{d}x \int \text{d}x \text{d}x \]

\[ \frac{f}{\text{area}(B_i-x_j)} \]

(all possible ways to partition a given genus surface)
\[ f_n(a) = \sum_{\text{all Feynman diagrams of genus } n} A \]

\[ A = \pi \int d\Omega_{\xi} \prod_{i,j} G(\xi_i - \xi_j) \]

Each Feynman diagram can be considered as a partition of the genus-\(n\) surface. (b) can be considered a dual diagram of (a).

\[ f_n(a) = \sum_{\text{sum over all possible tranquilaters}} \int d\Omega_{\phi} \phi e^{-Sp} \]

with some target space (related to propagator \( \mathcal{G} \))

In particular, in order to have a continuum limit, we need regulated diagrams with \( \phi \to 0 \).
large # of vertices to dominate.

\[ \Rightarrow \text{A large or near some singularity of } f(a) \] (directly)

\[ \text{If } G(x_i - A^i_j) \text{ to deduce } Sp(2\pi, \mathbb{R}) \text{ is impossible.} \]

2nd lower dimensions: \( D = 0, 1 \) this is possible indirectly.

P2: \( \text{not known} \) (at the understanding Liouville field)

= \[ \frac{1}{N} \] expansion of gauge theory \( \rightarrow \) some strong theory

but not clear how to find it.

(vacuum diagram level)
Now consider including external objects:

\[ \text{Each external object introduces a boundary on the 2d-world sheet} \]

For surfaces with boundaries:

\[ X = \frac{1}{\pi} \int_{\partial\Sigma} \mathcal{F} \wedge R = 2 - 2g - b \]

where \( b \) is the number of boundaries.
Now each surface is weighted by

\[ e^{-\lambda \phi (2-2\hbar-6) \Phi_{2\hbar-2+b}} \]

e.g. N-stry scattering: \[ \propto \Phi_{2\hbar} ^{n-2} (1 + q_2 + \ldots) \]

\[ \langle 0(x_1) \ldots 0_n(x_n) \rangle \sim N^{2-n} \]

We can identify our glue ball states

\[ C(x) \otimes 0(x_1) 0 \leftrightarrow \text{Closed string states} \]

\[ \text{One step Closed string states} \]

\[ \text{Classical theory of} \]

\[ \text{N=4, classical theory of glue balls} \leftrightarrow \text{Classical theory of strings} \]
Some generalizations:

1. On the gauge theory side, we can include fields in fundamental rep (quarks).

Vacuum diagrams: **now include loops of quarks**

Can now be classified topologically by 2d surfaces with boundaries

- A string theory with also open strings

2. Instead of $SU(N)$, consider $SO(N)$ or $Sp(N)$

\[
\begin{align*}
U(N) & \quad \langle A^a_{\mu} A^b_{\nu} \rangle = \frac{a}{b} \frac{d}{c} \\
SO(N) & \quad \langle A^a_{\mu} A^b_{\nu} A^c_{\rho} \rangle = \frac{a}{b} \frac{d}{c}
\end{align*}
\]

2d surface with no orientation

Non-orientable string
New take a gauge theory in \((3+1)\)-dimensions, Minkowski, whose \(1/N\) expansion should be described by a string theory.

What can we say about this string theory?

Simplest possibility: consider a string theory in \((3+1)\)-d Mink.

\[ ds^2 = -dt^2 + dx^2 = 2 J_{\mu \nu} dx^\mu dx^\nu \]

\[ S_p = \int \text{d}t \text{d}x \, F_{8} \wedge \text{d} x^8 \wedge \text{d} x^8 \wedge J_{\mu \nu} \]

but this does not work:

(1) such a string theory is in consistent \(D=10\)

(2) Take a \(6\)g in \(10\)D with \(M_4 \times M\) some compact manifold

\(=\) such a theory contains a massless \(spin-2\) particle (graviton), not present in YM theory!
Possible way out:  
1) give up
2) some more exotic string action
3) take the Liouville mode seriously.

local \( \phi \) not dynamical:  \( \text{Diff} \times \text{Weyl} \Rightarrow \delta \phi = \delta \phi \)

but if Weyl transformation becomes anomalous:  
\[ \delta \phi = e^\Phi \delta \phi \]
\( \Phi : \text{Liouville mode} \)
\( \text{can become dynamical} \)

\[ \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1} + \phi : \text{add one more dimension} \]

This idea appears to work for \( D=0,1 \) dimensions.

Not known how to do it for higher dimensions.

Polyakov (1997): treat \( \phi \) as an extra dimension
\[ \text{curved, uncompact} \]
\[ ds^2 = a^2(\phi) \left( d\phi^2 + \sum dx^i dx_i \right) \]
\[ \text{most general metric} \]
This idea is very appealing: avoids previous difficulties with gravity: described by some theory with 1 less dimension.

Also: holographic principle: QG \rightarrow 6d theory in 1 higher dimension.

For a general XM theory, not much can be said about O, but for a scale-invariant theory, symmetries can constrain O more. => AdS:

A scale-invariant theory is invariant under

\[ x^\mu \rightarrow \lambda x^\mu, \quad \lambda: \text{any constant} \]

O should also be invariant under such a scale.
\[ \phi \to \lambda \phi \]

\[ a(\phi) \to \frac{a(\phi)}{\lambda} \]

\[ a(\phi) = \frac{R^0}{\phi^2} \quad R: \text{some constant} \]

\[ ds^2 = \frac{R^2}{\phi^2} \left( d\phi^2 + \delta_{ij} dx^i dx^j \right) \]

AdS spacetime!

R: curvature radius

Confidential field theory

in \( d \) dimensional spacetime

\[ \leftrightarrow \]

\[ \leftrightarrow \]

a string theory

in AdS \( d+1 \)

\[ N^{-1} \leftrightarrow \quad \theta \leftrightarrow g_s \]

\[ \lambda \leftrightarrow f\left( \frac{R^2}{\phi^2} \right) \]
Expect large $\lambda$ should correspond to small curvature, i.e., $\frac{\lambda^2}{\kappa}$ large.

Now: in principle one can develop what we know about AdS/CFT from here. (e.g., work out a dictionary.)

In reality, Polyakov stopped at 0.

But an explicit example is powerful!

A few months later, Maldacena discussed some explicit examples that we now call AdS/CFT correspondence from a completely different route with some explicit examples.