10 Suppose a square has 9 nodes—its corners, edge midpoints, and center. The rows of $K$ will have varying numbers of nonzeros (a varying bandwidth):

(a) How many nodes are neighbors of a corner node? They lie in the four squares that meet at the corner.

(b) How many nodes are neighbors of a midpoint node?

(c) For the node at the center of the square, why is elimination permitted within the element matrix $K_e$ before assembling it with others into $K$?

11 (a) If $U = a + bx + cy + dxy$ find the coefficients $a, b, c, d$ from the four equations $U = U_1, U_2, U_3, U_4$ at the corners $(\pm 1, \pm 1)$ of a standard square.

(b) From part (a) write down the 2 by 4 gradient matrix $G$ at the center point $P = (0,0)$. The derivatives $b + dy$ and $c + dx$ reduce there to $b$ and $c$:

At the center

\[
\begin{bmatrix}
\frac{\partial U}{\partial x} \\
\frac{\partial U}{\partial y}
\end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} = G \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}
\]

(c) Show that $(1, 1, 1, 1)$ and $(1, -1, 1, -1)$ are in the nullspace of $G$. The first comes from a constant $U = 1$ and correctly has zero energy. The second comes from an "hourglass" $U = xy$ and should have positive energy.

12 From the previous problem find the approximation $K_e = 4G^T G$ for the $Q_1$ bilinear element on the square of area 4. Compare with the correct $K_e$ found by integrating $(b + dy)^2 + (c + dx)^2$ analytically.

13 If $U = a + bx + cy$ on the triangle below, find $b$ and $c$ from the equations $U = U_1$, $U = U_2$, $U = U_3$ at the nodes. Show that the gradient matrix $G$ is

\[
\begin{bmatrix} b \\ c \end{bmatrix} = G U = \begin{bmatrix}
\frac{1}{L} & \frac{1}{L} & 0 \\
\frac{d}{Lh} & \frac{1}{h} & \frac{d}{Lh} & \frac{1}{h}
\end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}.
\]

14 (a) For the previous matrix multiply $G^T G$ by the area $Lh/2$ to find the element stiffness matrix $K_e$. For the standard triangle $T$ reduce it to (28).
Problem Set 4.1

1. Find the Fourier series on $-\pi \leq x \leq \pi$ for
   
   (a) $f(x) = \sin^3 x$, an odd function
   
   (b) $f(x) = |\sin x|$, an even function
   
   (c) $f(x) = x$
   
   (d) $f(x) = e^x$, using the complex form of the series.

   What are the even and odd parts of $f(x) = e^x$ and $f(x) = e^{ix}$?

2. From Parseval’s formula the square wave sine coefficients satisfy

   $$\pi (b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi.$$ 

   Derive the remarkable sum $\pi^2 = 8\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right)$.

3. If a square pulse is centered at $x = 0$ to give

   $$f(x) = 1 \quad \text{for} \quad |x| < \frac{\pi}{2}, \quad f(x) = 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi,$$

   draw its graph and find its Fourier coefficients $a_k$ and $b_k$.

4. Suppose $f$ has period $T$ instead of $2x$, so that $f(x) = f(x + T)$. Its graph from $-T/2$ to $T/2$ is repeated on each successive interval and its real and complex Fourier series are

   $$f(x) = a_0 + a_1 \cos \frac{2\pi x}{T} + b_1 \sin \frac{2\pi x}{T} + \cdots = \sum_{-\infty}^{\infty} c_k e^{ik2\pi x/T}$$

   Multiplying by the right functions and integrating from $-T/2$ to $T/2$, find $a_k, b_k, \text{and} c_k$.

5. Plot the first three partial sums and the function itself:

   $$x(\pi - x) = \frac{8}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots \right), \quad 0 < x < \pi.$$ 

   Why is $1/k^3$ the decay rate for this function? What is the second derivative?

6. What constant function is closest in the least square sense to $f = \cos^2 x$? What multiple of $\cos x$ is closest to $f = \cos^3 x$?

7. Sketch the $2\pi$-periodic half wave with $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = 0$ for $-\pi < x < 0$. Find its Fourier series.

8. (a) Find the lengths of the vectors $u = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ and $v = (1, \frac{1}{3}, \frac{1}{9}, \ldots)$ in Hilbert space and test the Schwarz inequality $|u^T v|^2 \leq (u^T u)(v^T v)$. 
Exactly the same idea gives a fast inverse transform. The matrix $F_N^{-1}$ contains powers of the conjugate $\bar{w}$. We just replace $w$ by $\bar{w}$ in the diagonal matrix $D$, and in formula (16). At the end, divide by $N$.

One last note about this remarkable algorithm. There is an amazing rule for the order that the $c$’s enter the butterflies, after all $L$ of the odd-even permutations. Write the numbers 0 to $N - 1$ in base 2. Reverse the order of their bits (binary digits). The complete flow graph shows the bit-reversed order at the start, then $L = \log_2 N$ recursion steps. The final output is $F_N$ times $c$.

The fastest FFT will be adapted to the processor and cache capacities of each specific computer. There will naturally be differences from a textbook description, but the idea of recursion is still crucial. For free software that automatically adjusts, we highly recommend the website fftw.org.

Problem Set 4.3

1. Multiply the three matrices in equation (14) and compare with $F$. In which six entries do you need to know that $i^2 = -1$? This is $(w_4)^2 = w_2$. If $M = N/2$, why is $(w_N)^M = -1$?

2. Why is row $i$ of $\bar{F}$ the same as row $N - i$ of $F$ (numbered from 0 to $N - 1$)?

3. From Problem 2, find the 4 by 4 permutation matrix $P$ so that $F = P \bar{F}$. Check that $P^2 = I$ so that $P = P^{-1}$. Then from $\bar{F}F = 4I$ show that $P = F^2/4$. It is amazing that $P^2 = F^4/16 = I$! Four transforms of $c$ bring back 16 $c$.

Note For all $N$, $F^2/N$ is a symmetric permutation matrix $P$. It has the rows of $I$ in the order 1, $N$, $N - 1$, ..., 2. Then $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = I([1, N : -1 : 2], :)$ for the reverse identity $I$. From $P^2 = I$ we find (surprisingly!) that $F^4 = N^2 I$. The key facts about $P$ and $F$ and their eigenvalues are on the cse website.

4. Invert the three factors in equation (14) to find a fast factorization of $F^{-1}$.

5. $F$ is symmetric. Transpose equation (14) to find a new Fast Fourier Transform!

6. All entries in the factorization of $F_6$ involve powers of $w =$ sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 \\ F_3 \end{bmatrix} \begin{bmatrix} P \end{bmatrix}.$$

Write down these factors with 1, $w$, $w^2$ in $D$ and powers of $w^2$ in $F_3$. Multiply!

7. By analogy with the discrete sine $(0, 1, 0, -1)$ what is the discrete cosine vector $(N = 4)$? What is its transform?

8. Put the vector $c = (1, 0, 1, 0)$ through the three steps of the FFT to find $y = Fc$. Do the same for $c = (0, 1, 0, 1)$.

9. Compute $y = F_8 c$ by the three FFT steps for $c = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation for $c = (0, 1, 0, 1, 0, 1, 0, 1)$. 
1. **When you multiply numbers you are convolving their digits.** We have to "carry" numbers in actual multiplication, while convolution leaves them in the same decimal place. What is $t$?

$$(12)(15) = (180) \quad \text{but} \quad (\ldots, 1, 2, \ldots) \ast (\ldots, 1, 5, \ldots) = (\ldots, 1, 7, t, \ldots).$$

2. Check the cyclic convolution rule $F(c \ast d) = (Fc) \ast (Fd)$ directly for $N = 2$:

$$F = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad Fc = \begin{bmatrix} c_0 + c_1 \\ c_0 - c_1 \end{bmatrix}, \quad Fd = \begin{bmatrix} d_0 + d_1 \\ d_0 - d_1 \end{bmatrix}, \quad c \ast d = \begin{bmatrix} c_0d_0 + c_1d_1 \\ c_0d_1 + c_1d_0 \end{bmatrix}.$$ 

3. Factor the 2 by 2 circulant $C = \begin{bmatrix} c_0 & c_1 \\ c_1 & c_0 \end{bmatrix}$ into $F^{-1} \text{diag}(Fc)F$ from Problem 2.

4. The right side of (12) shows the fast way to convolve three fast transforms will compute $Fc$ and $Fd$ and transform back by $F^{-1}$. For $N = 128, 1024, 8192$ create random vectors $c$ and $d$. Compare tic; cconv($c$, $d$); toc; with this FFT way.

5. Write the steps to prove the Cyclic Convolution Rule (13) following this outline: $F(c \ast d)$ has entries $\sum (\sum c_n d_{k-n}) w^{jk}$. The inner sum on $n$ produces $c \ast d$ and the outer sum on $k$ multiplies by $F$. Write $w^{jk}$ as $w^{jn}$ times $w^{j(k-n)}$. When you sum first on $k$ and last on $n$, the double sum splits into $\sum c_n w^{jn} \sum d_k w^{jk}$.

6. What is the identity vector $\delta_N$ in cyclic convolution? It gives $\delta_N \ast d = d$.

7. Which vectors $s$ and $s_N$ give one-step delays, noncyclic and cyclic?

$s \ast (\ldots, d_0, 1, \ldots) = (\ldots, d_1, 0, \ldots)$ and $s_N \ast (d_0, \ldots, d_{N-1}) = (d_{N-1}, d_0, \ldots)$.

8. (a) Compute directly the convolution $f \ast f$ (cyclic convolution with $N = 6$) when $f = (0, 0, 0, 1, 0, 0)$. Connect $(f_0, \ldots, f_5)$ with $f_0 + f_1 w + \cdots + f_5 w^5$.

(b) What is the Discrete Transform $c = (c_0, c_1, c_2, c_3, c_4, c_5)$ of this $f$?

(c) Compute $f \ast f$ by using $c$ in "transform space" and transforming back.

9. Multiplying $C_{\infty}D_{\infty}$ will convolve $c \ast d$ and multiply $(\sum c_k e^{ikx})(\sum d_k e^{ikx})$. If $D = C^T$ is real, this is an autocorrelation of $c$ leading to $|c_k e^{ikx}|^2 > 0$. For $c = 1, 3, 4$ show directly that the diagonals of $CC^T = C^TC$ (positive definite) agree with $(1, 3, 4) \ast (4, 3, 1) = (4, 10, 21, 10, 4)$ in equation (8).

10. The chance of grade $i = (70, 80, 90, 100)$ on one quiz is $p = (0.3, 0.4, 0.2, 0.1)$. What are the probabilities $c_k$ for the sum of two grades to be $k = (140, 150, \ldots, 200)$? You need to convolve $c = p \ast p$ or multiply 3421 by 3421 (without carrying).

11. What is the expected value (mean $m$) for the grade on that quiz? The generating function is $P(z) = .3z^{70} + .4z^{80} + .2z^{90} + .1z^{100}$. Show that $m = p'(1)$. 